Regular or stochastic dynamics in families of higher-degree unimodal maps

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Regular or stochastic dynamics in families of higher-degree unimodal maps

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Abstract. We construct a lamination of the space of unimodal maps with critical points of fixed degree $d \geq 2$ by the hybrid classes. The structure of the lamination yields a partition of the parameter space for one-parameter real analytic families of unimodal maps and allows us to transfer *a priori* bounds in the phase space to the parameter space. This implies that almost every map in such a family is either regular or stochastic.

Contents

1. *Introduction*

In [**25**], Palis conjectured that 'typical' dissipative dynamical systems can be described by finitely many attractors, each supporting an ergodic physical measure, and that this description is robust under sufficiently random perturbations of the system. In a series of papers, [**2, 4–7, 20, 22**], the authors answer this question in full for generic families of unimodal maps. Central to their results is the fact that a generic unimodal map f has a non-degenerate critical point. This condition guarantees that landing maps associated to certain high iterates of some unimodal maps f are almost linear. In this paper, we consider families of maps where the critical point has fixed even degree $d \geq 2$. Our goal is to prove the main theorems of [**2, 6**] in this setting.

We say that a map of the interval is *regular* if its critical orbit converges to an attracting cycle, and *non-regular* otherwise. If f is a non-regular map, we define its *real hybrid class* to be its topological conjugacy class. If f is regular, we need to refine its topological conjugacy class to obtain its real hybrid class. In the case of a map f with negative Schwarzian derivative, we restrict the topological conjugacy class to those maps with the property that the multiplier of the attracting periodic orbit that contains the critical point in its immediate basin of attraction is the same as that of f, and in the general case we use the refinement of [**6**, Appendix A]. We say that a family of maps is *non-trivial* if it is not contained in a single hybrid class. The following theorem about the structure of the space of unimodal maps is instrumental in the measure-theoretic study of analytic families of maps.

THEOREM A. *Every real hybrid class, $\mathcal{H}_f^{\mathbb{R}}$, is an embedded codimension-one real analytic submanifold of $\mathcal{U}_a$. Furthermore, the hybrid classes laminate a neighbourhood of every map without parabolic cycles.*

The space, $\mathcal{U}_a$, is a space of unimodal maps and will be described precisely later.

In the proof of this theorem, the main new difficulty we encounter is the presence of at most finitely renormalizable maps without decay of geometry, a phenomenon that does not occur for maps with a quadratic critical point; however, these maps possess generalized polynomial-like generalized renormalizations and can be treated in certain circumstances almost as if they are infinitely renormalizable. Once Theorem A is proved, we proceed by using the local lamination structure and properties of a generalized renormalization operator to partition the family according to the combinatorics of the critical point and transfer certain geometric bounds in the dynamical plane for at most finitely renormalizable maps to the parameter space. This puts us in a position to apply statistical arguments of [**2, 3, 5, 6**] to prove the following theorem.

THEOREM B. *Suppose that $\{f_\lambda\}$ is a non-trivial analytic family of unimodal maps, and that the set of infinitely renormalizable maps in the family has measure zero. Then almost any map in the family is regular or stochastic.*

Remark 1.1. It has recently been announced that the renormalization operator acting on the space of degree d unicritical polynomial-like maps is hyperbolic [**1**], which implies that in the families we considering the sets of infinitely renormalizable maps have measure zero.

1.1. *Outline of the proof.* We begin by proving Theorem A. For maps with simple combinatorics, hyperbolic and Misiurewicz maps, the arguments are identical to those in the quadratic case, so we will spend little time on them. We refer the reader to [**2**] and [**6**, Appendix A] for the proof of Theorem A in these cases.

The remaining maps are those whose critical point is recurrent. We will assume throughout that 0 is the critical point. We consider the following two cases separately: the maps for which $\omega(0)$ is minimal, and those for which the geometry is sufficiently big (there is a nice interval J containing the critical point such that $|J|/|J_0|$ is very large, where J_0 is the component of the domain of the return map to J containing the critical point). For such a map f with sufficiently big geometry, the proof in the case when f is quadratic that the hybrid class of f is an embedded codimension-one submanifold in $\mathcal{U}_a$ requires only a few changes in the higher-degree case. If f is a map for which the $\omega(0)$

is minimal, we use results that appeared first in [**15, 17**], and were later generalized in [**27**], to construct a puzzle map given by a complexification of the first landing map under f to some sufficiently small nice interval containing the critical point, and we show that certain restrictions of these puzzle maps persist under small perturbations of our map. We also show that an appropriate restriction of the return map is a generalized polynomial-like map.

The use of renormalization arguments is particularly well suited to the study of the infinitesimal structure of the space of unimodal maps. In [**21**], Lyubich endows the space of polynomial-like maps with a complex analytic structure modelled on a family of Banach balls, and uses it to study the space of polynomial-like maps. Parts of this theory were first generalized to generalized polynomial-like maps in [**29**]. Once we have proven the necessary results in the space of generalized polynomial-like maps, we use the renormalization operator to pull them back to the space of unimodal maps. In order to construct the lamination, we need to carry out certain approximation arguments in the space of unimodal maps. To do this, we require the existence of the persistent puzzle map mentioned above. Even though the puzzles we construct for maps with minimal post-critical set in general do not have the good geometric properties of those that can be constructed for maps with sufficiently big geometry, the fact that they are persistent is sufficient for our purposes.

Once we have constructed the lamination we will be able to partition the parameter space according to combinatorics of the return map of the critical point back to deep levels of the principal nest and thus construct the principal nest of ‘parapuzzle pieces’. Moreover, we will be in a position to apply arguments of [**6**] to show that for parameters with sufficiently big geometry this nest has *a priori bounds*. For maps without sufficiently big geometry, we use arguments that rely on the control of the geometry of certain puzzle pieces in the dynamical plane provided by the enhanced nest of [**13**] and the analyticity of the generalized renormalization operator to transfer *a priori* bounds in the parameter space for families of generalized polynomial-like maps back to the family under consideration. With the *a priori* bounds theorem proved, we can use a parameter exclusion argument of [**3**] to show that the set of parameters with exponential decay of geometry in the dynamical plane, decay of geometry in the principal nest of parapuzzle pieces, and whose principal nest is eventually free of central returns has full measure in the set of non-regular parameters. Then the arguments of [**5, 6**] complete the proof (see [**5**, Remark 2.1]).

2. *Preliminaries*

2.1. *General notation.* We let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\overline{\mathbb{C}}$ denote the natural numbers, the real line, the complex plane and the Riemann sphere, respectively. We will let I be the interval $[-1, 1]$. If $J = (a - x, a + x)$ is any interval, for $\eta > 0$ we define $\eta J = (a - \eta x, a + \eta x)$. For $r > 0$, we let $\mathbb{D}_r \subset \mathbb{C}$ denote the disc of radius r and we let $\mathbb{D} = \mathbb{D}_1$. If $r > 1$, we let $A_r = \{z \in \mathbb{C} : 1 < |z| < r\}$. If $A \subset \mathbb{C}$ is any annulus, then A is uniformized by some unique A_r. If A is uniformized by A_r, we denote the *modulus* of A by $\mathrm{mod}(A) = \ln(r)$. Given $a > 0$, we let $\Omega_a = \{z \in \mathbb{C} : \mathrm{dist}(z, I) < a\}$. If $V \subset \mathbb{C}$, we let $\mathrm{Comp}_x(V)$ denote the connected component of V containing x.

Suppose that $f : U \to U'$ where $U, U' \subset \mathbb{C}$. If $V \subset U$ we let R_V denote the first return map to V under f whenever it is defined. If $x \in U$ has the property that for some $m > 0$, $f^m(x) \in V$, we define $\mathcal{L}_x(V) = \mathrm{Comp}_x(f^{-n}(V))$, where $n > 0$ is minimal so that $f^n(x) \in V$.

For any point x in the domain of f we let $\mathrm{orb}_f(x) = \{f^i(x)\}_{i=0}^{\infty}$. We also use this notation for partially defined maps in which case $\mathrm{orb}_f(x)$ consists of those points $f^n(x)$ that are well defined.

Suppose that $\mathcal{A}$ is a Banach space. If $\Lambda \subset \mathcal{A}$, a *graph* of a continuous map $h : \Lambda \to \mathbb{C}$ is $\{(\lambda, h(\lambda)) \in \mathcal{A} \oplus \mathbb{C} : \lambda \in \Lambda\}$.

We define $\mathbf{0} : \mathcal{A} \to \mathcal{A} \oplus \mathbb{C}$ by $\mathbf{0}(\lambda) = (\lambda, 0)$.

Let $\pi_1 : \mathcal{A} \oplus \mathbb{C} \to \mathcal{A}$, $\pi_2 : \mathcal{A} \oplus \mathbb{C} \to \mathbb{C}$ be the coordinate projections. Given a set $\chi \subset \mathcal{A} \oplus \mathbb{C}$, we denote its fibres by $X[\lambda] = \pi_2(\chi \cap \pi_1^{-1}(\lambda))$.

A *fibrewise map* $\mathbf{F} : \chi \to \mathcal{A} \oplus \mathbb{C}$ is a map such that $\pi_1 \circ \mathbf{F} = \pi_1$. We denote the fibres of the map by $F[\lambda] : X[\lambda] \to \mathbb{C}$, so that $\mathbf{F}(\lambda, z) = (\lambda, F[\lambda](z))$.

We let $B_r(\mathcal{A})$ denote the ball of radius r about 0 in $\mathcal{A}$.

We say that a domain $\Lambda \subset \mathbb{C}$ is hyperbolic if its complement contains at least three points. If Λ is a hyperbolic domain, its universal covering space is conformally equivalent to $\mathbb{D}$, so such a domain can be equipped with a hyperbolic metric, d_Λ, obtained by pushing down the Poincaré metric on the unit disc to Λ. If $x, y \in \Lambda$, a hyperbolic domain, we let $\mathrm{dist}_\Lambda(x, y)$ denote the distance between x and y measured in the hyperbolic metric on Λ.

We will say that a Banach space $\mathcal{A}$ is the complexification of a real-symmetric Banach space $\mathcal{A}^{\mathbb{R}}$ if there exists an anti-linear isometric involution, denoted **conj**, fixing $\mathcal{A}^{\mathbb{R}}$. We extend **conj** to $\mathcal{A} \oplus \mathbb{C}$ by $\mathbf{conj} : (\lambda, z) \mapsto (\mathbf{conj}(\lambda), \overline{z})$. A set $X \subset \mathcal{A}$ or $\mathcal{A} \oplus \mathbb{C}$ is called *real-symmetric* if $\mathbf{conj}(X) = X$.

2.2. *Quasiconformal and quasisymmetric maps.* Let $U \subset \mathbb{C}$ be a domain. A map $h : U \to \mathbb{C}$ is K-quasiconformal (K-qc) if it is a homeomorphism onto its image and if for any annulus $A \subset U$, $\mathrm{mod}(A)/K \le \mathrm{mod}(h(A)) \le K\mathrm{mod}(A)$. The minimum such K is the *dilatation* of h, denoted by $\mathrm{Dil}(h)$.

We say that a topological disc is a *quasidisc* if it is the image of $\mathbb{D}$ under a quasiconformal mapping of $\mathbb{C}$.

A closed set $A \subset \mathbb{C}$ is qc *removable* if any qc map defined on $\mathbb{C}\backslash A$ extends uniquely to a qc map on $\mathbb{C}$.

To any quasiconformal map $h : U \to \mathbb{C}$, one may associate the *Beltrami differential of* h,

$$\mu_h = \frac{\overline{\partial} h}{\partial h}\frac{d\overline{z}}{dz},$$

with $\|\mu_h\|_\infty < 1$.

A homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is said to be γ-quasisymmetric (γ-qs) if it has a γ-qc, real-symmetric extension $\tilde{h} : \mathbb{C} \to \mathbb{C}$. If $X \subset \mathbb{R}$, we will say that $h : X \to \mathbb{R}$ is γ-qs if it has a γ-qs extension to $\mathbb{R}$.

2.3. *Spaces of maps.* If U is a domain in the complex plane, let $\mathcal{B}(U)$ be the space of bounded holomorphic functions on U. Let $a > 0$. Let $\mathcal{E}_a \subset \mathcal{B}(\Omega_a)$ be the complex

Banach space of holomorphic maps $v : \Omega_a \to \mathbb{C}$ that are continuous up to the boundary, 0-symmetric and such that $v(-1) = v(1) = 0$, endowed with the sup norm $\|v\|_a = \|v\|_\infty$. It contains the real Banach space $\mathcal{E}_a^{\mathbb{R}}$ of real maps; that is, the subspace of $\mathcal{E}_a$ consisting of those maps that are $\mathbb{R}$-symmetric.

Let $-1 \in \Omega_a$ be the constant function. We let $\mathcal{A}_a$ denote the complex affine subspace $-1 + \mathcal{E}_a$.

2.4. *Holomorphic motions and laminations.* Let Λ be a connected open subset of a Banach space $\mathcal{A}$. A *holomorphic motion* h over Λ is a family of holomorphic maps defined on Λ whose graphs, $\{h_\lambda(z) : \lambda \in \Lambda\}$, do not intersect. We will call such a graph the *leaf* of the holomorphic motion through z. The *support* of h is the set $\boldsymbol{\chi} \subset \mathbb{C}^2$, the union of the leaves of h.

The *holonomy maps*, $h[\lambda, \lambda'] : X[\lambda] \to X[\lambda']$, where $\lambda, \lambda' \in \Lambda$, of a holomorphic motion are defined by $h[\lambda, \lambda'](z) = y$ if (λ, z) and (λ', y) belong to the same leaf.

Given a holomorphic motion h over a domain Λ, a holomorphic motion h' over a domain $\Lambda' \subset \Lambda$ whose leaves are contained in the leaves of h is called a *restriction* of h. If h' is a restriction of h, we say that h is an *extension* of h'.

Define a function $K : [0, 1) \to \mathbb{R}$ by $K(r) = (1+\rho)/(1-\rho)$, where $0 \le \rho < 1$ is such that the hyperbolic distance in $\mathbb{D}$ between 0 and ρ is r.

LEMMA 2.1. (λ-lemma [**8, 23**]) *Let h be a holomorphic motion over a hyperbolic domain in $\mathbb{C}$ and let $\lambda, \lambda' \in \Lambda$. Then $h[\lambda, \lambda']$ extends to a qc map of $\mathbb{C}$, with dilatation bounded by* $K(\mathrm{dist}_\Lambda(\lambda, \lambda'))$.

If Λ is not one-dimensional, the same estimate holds with the Kobayashi distance instead of the hyperbolic distance. In particular, if h is a holomorphic motion over $B_r(\mathcal{A})$, and if $\lambda, \lambda' \in B_{r/2}(\mathcal{A})$, then $h[\lambda, \lambda'] = 1 + O(\|\lambda - \lambda'\|)$.

If h is the holomorphic motion of an open set, we define $\mathrm{Dil}(h)$ as the supremum of the dilatations of the holonomy maps, $h[\lambda, \lambda']$.

Suppose that Λ is a real-symmetric domain. A holomorphic motion h over Λ is called real-symmetric if the image of any leaf under **conj** is a leaf.

LEMMA 2.2. (Extension lemma [**8, 28**])
(1) *If h is a holomorphic motion over a simply connected domain $\Lambda \subset \mathbb{C}$, then there exists an extension of h to a holomorphic motion of $\mathbb{C}$ over Λ.*
(2) *If h is a holomorphic motion over $B_r(\mathcal{A})$, then the restriction of h to $B_{r/3}(\mathcal{A})$ can be extended to a holomorphic motion of $\mathbb{C}$ in a canonical way.*

Moreover, in either case, if h is a real-symmetric holomorphic motion, then the extension of h given by either (1) *or* (2) *can be taken to be real-symmetric.*

From now on, we will always assume that extensions of holomorphic motions are real-symmetric.

Let $\mathcal{A}$ be a Banach space. A *codimension-one holomorphic lamination* $\mathcal{L}$ on an open set $B \subset \mathcal{A}$ is a family of disjoint codimension-one Banach submanifolds of $\mathcal{A}$, called the *leaves* of the lamination, such that for any point $p \in B$, there exists a holomorphic local chart $\phi : \tilde{B} \to \mathcal{V} \oplus \mathbb{C}$, where $\tilde{B} \subset B$ is a neighbourhood of p and $\mathcal{V}$ is an open set in some complex Banach space $\tilde{\mathcal{A}}$, with the property that for any leaf L and any connected component L_0 of $L \cap B$, the image $\phi(L_0)$ is the graph of a holomorphic function $\mathcal{V} \to \mathbb{C}$. It is useful to notice that the local theory of codimension-one laminations is the theory of holomorphic motions. For instance, we have the following lemma.

LEMMA 2.3. *Any codimension-one holomorphic lamination is transversally quasiconformal.*

In other words, the holonomy maps of the lamination are qc, have qc-extensions to holomorphic motions of $\mathbb{C}$ and the λ-lemma gives bounds on the dilatations of the extensions.

2.5. *Tubes and tube maps.* The terminology and notation given below was introduced in [**6**], and we will follow their exposition closely. It will make it possible to present our later results in a concise fashion.

A *proper motion* of a set X over Λ is a holomorphic motion of X over Λ with support $\boldsymbol{\chi}$ such that for any $\lambda \in \Lambda$, the map $h : \Lambda \times X[\lambda] \to \boldsymbol{\chi}$ defined by $h[\lambda](\lambda', x) = (\lambda', h[\lambda, \lambda'](x))$ has an extension to $\overline{\Lambda} \times X[z]$ that is a homeomorphism. An *equipped tube* h_T is a holomorphic motion of a Jordan curve T. Its support is called a *tube*; we say that an equipped tube is *proper* if it is a proper motion, and in this case its support is called a *proper tube*. The *filling* of a tube $\mathbf{T}$ is the set $\mathbf{U} \subset \Lambda \times \mathbb{C}$ such that $U[\lambda]$ is the bounded component of $\mathbb{C} \backslash T[\lambda]$, $\lambda \in \Lambda$. A *special motion* is a holomorphic motion h of $X \cup T$ such that its support, $\boldsymbol{\chi}$, is contained in the filling $\mathbf{U}$ of $\mathbf{T}$, $h|T$ is an equipped proper tube and the closure of any leaf through X does not intersect the closure of $\mathbf{T}$. If $\mathbf{T}$ is a tube over Λ and $\mathbf{U}$ is its filling, a fibrewise holomorphic map $\mathbf{F} : \mathbf{U} \to \mathbb{C}^2$ is called a *tube map* if it admits a continuous extension to $\overline{\mathbf{U}}$.

Let $\mathbf{F} : \mathbf{V} \to \mathbb{C}^2$ be a tube map such that $\mathbf{F}(\partial \mathbf{V}) = \partial \mathbf{U}$, where $\mathbf{U}$ is the filling of a tube over Λ, and let h be a holomorphic motion supported on $\overline{\mathbf{U}} \cap \pi_1^{-1}(\Lambda)$. Let Γ be an open set such that $\overline{\Gamma} \subset \Lambda$ and let W be an open set that moves holomorphically by h over Λ with $\overline{W} \subset U$ and such that $\overline{W}$ contains the critical values of $\mathbf{F}|(\mathbf{V} \cap \pi_1^{-1}(\overline{\Gamma}))$. Consider now a leaf of h through $z \in U \backslash \overline{W}$ and let $\mathbf{E}(z)$ denote its preimage under $\mathbf{F}$ intersected with $\pi_1^{-1}(\Gamma)$. Each connected component of $\mathbf{E}(z)$ is a graph over Γ. Moreover, $\overline{\mathbf{E}}(z) \subset \mathbf{U}$, so the set of connected components of $\mathbf{E}(z)$, where $z \in U \backslash \overline{W}$, is a holomorphic motion over Γ. We define a new holomorphic motion over Γ, called the *lift* of h by $(\mathbf{F}, \Gamma, W)$, as an extension to the closure of V of the holomorphic motion whose leaves are the connected components of $\mathbf{E}(z)$ for $z \in U \backslash \overline{W}$. This holomorphic motion is a special motion of V over Γ and its dilatation is over $F^{-1}(U \backslash \overline{W})$ is bounded by $K(r)$, where r is the hyperbolic diameter of Γ in Λ.

Let h be an equipped proper tube supported on $\mathbf{T}$. A *diagonal* of $\mathbf{T}$ is a holomorphic section $\Psi : \Lambda \to \mathbb{C}^2$ admitting a continuous extension to Λ and such that $\Psi(\Lambda)$ is contained in the filling of $\mathbf{T}$ and, for $\lambda \in \Lambda$, $h[\lambda] \circ \Psi|\partial\Lambda$ is a degree-one map onto $T[\lambda]$. Let h be a special motion of $X \cup T$ and let ϕ be a diagonal of $h|T$. The argument principle (see [**20**]) implies that the leaves of $h|X$ intersect $\phi(\Lambda)$ in a unique point with multiplicity one. So we can define a map $\chi[\lambda] : X[\lambda] \to \Lambda$ such that $\chi[\lambda](z) = w$ if (λ, z) and $\phi(w)$ belong to the same leaf of h. Each $\chi[\lambda]$ is a homeomorphism onto its image; moreover, if $U \subset X$ is open, $\chi[\lambda]|U[\lambda]$ is locally quasiconformal and $\mathrm{Dil}(h|U) < \infty$, then $\chi[\lambda]|U[\lambda]$ is globally quasiconformal with dilatation bounded by $\mathrm{Dil}(h|U)$. We will call χ the *holonomy family* associated to the pair (h, ϕ).

2.6. *Quasiconformal vector fields.* A continuous vector field $\alpha = \alpha(z)/dz$ on an open set $U \subset \overline{\mathbb{C}}$ is called K-*quasiconformal* if it has locally integrable distributional derivatives $\partial\alpha$ and $\overline{\partial}\alpha$ and $\|\overline{\partial}\alpha\| \leq K$. A vector field is called *quasiconformal* if it is K-qc for some K.

Given $\mu \in L^{\infty}(\mathbb{C})$, one obtains a qc vector field α satisfying $\overline{\partial}\alpha = \mu$ locally via the Cauchy transform

$$-\frac{1}{\pi}\int \frac{\mu(\zeta)}{z-\zeta} d\zeta \wedge d\overline{\zeta}.$$

The Cauchy transform implies that the local solutions have modulus of continuity $\phi(x) = -x\ln(x)$.

LEMMA 2.4. (Compactness lemma for quasiconformal vector fields) *The space of K-qc vector fields of the Riemann sphere, $\overline{\mathbb{C}}$, vanishing at three given points is compact in the topology of uniform convergence on $\overline{\mathbb{C}}$.*

A continuous vector field v on a closed set $X \subset \mathbb{C}$ is called quasiconformal if it extends to a qc vector field on $\mathbb{C}$. If a vector field admits a normalized qc extension to $\overline{\mathbb{C}}$, then we let

$$\|v\|_{\mathrm{qc}} = \inf \|\overline{\partial}\beta\|_{\infty},$$

where β runs over all normalized qc extensions of v.

A K-qc vector field will be called normalized if it vanishes at $\{2, -2, \infty\}$.

Quasiconformal vector fields are tangent at the identity to holomorphic motions, making them the infinitesimal counterpart to qc maps.

LEMMA 2.5. [**2**, Lemma 2.10] *Let $h_\lambda : X \to \mathbb{C}$, $\lambda \in \mathbb{D}$, be a holomorphic motion with base point* 0*. Then*

$$\alpha \equiv \frac{d}{d\lambda} h_\lambda \bigg|_{\lambda=0}$$

is a qc vector field on X. Moreover, if X is an open set,

$$\overline{\partial}\alpha = \frac{d}{d\lambda}\mu_{h_\lambda}\bigg|_{\lambda=0}.$$

Let $f : \Omega \to \mathbb{C}$ be a holomorphic map and let v be a holomorphic vector field on Ω. A vector field α is called *equivariant* on a set $X \subset \Omega$ with respect to the pair (f, v) if for any $z \in X$,

$$v(z) = \alpha(f(z)) - f'(z)\alpha(z).$$

This equation can be rewritten as

$$f^*(\alpha) - \alpha = \frac{v}{f'}. \tag{2.1}$$

Let $X \subset \Omega$ and let α be a vector field on $Y \equiv f(X)$. A vector field β is a called the *lift* of α to X by (f, v) if $v = \alpha \circ f - f'\beta$.

Note that lifts preserve the qc-norm of vector fields. Assume that the set X is open and that the vector field α is quasiconformal. Let $Y \equiv f(X)$. The lift β of α by (f, v) can be written as $\beta = f^*\alpha - v/f'$, where v/f' is holomorphic. So,

$$\overline{\partial}\beta = \overline{\partial}(f^*(\alpha|Y)) = f^*\overline{\partial}(\alpha|Y),$$

where the first pullback acts on vector fields and the second acts on Beltrami differentials. Thus,

$$\|\overline{\partial}\beta\|_\infty = \|f^*\overline{\partial}(\alpha|Y)\|_\infty = \|\overline{\partial}(\alpha|Y)\|_\infty. \tag{2.2}$$

2.7. *Unimodal maps* A smooth map of the interval $f : I \to I$ is called *unimodal* if it has a single critical point of even degree, and this point is an extremum. We let $\omega(x)$ be the ω-limit set of a point x. Let $\mathcal{U}^3$ be the space of C^3 unimodal maps $f : I \to I$ with degree d critical point that are even symmetric (so that $f(-x) = f(x)$) and endow $\mathcal{U}^3$ with the C^3 topology. We will normalize the maps so that -1 is a fixed point and $f(1) = -1$. Moreover, we will assume that $Df(-1) \geq 1$ for otherwise the map has -1 as a global attractor or the map has a proper unimodal restriction to $[f(1), 1]$ (see [**9**]) and a stable fixed point in $[-1, f(1)]$. While it is not needed for everything we say, we restrict our attention to C^3 maps because certain restrictions of their high iterates enjoy many of the properties of maps with negative Schwarzian derivative (see [**10, 11**]). Let $\mathcal{U}_a = \mathcal{U}^3 \cap \mathcal{A}_a$.

Remark 2.1. The normalization and symmetry assumptions are made purely for convenience; indeed, through the argument of [**2**, Appendix C], all proofs generalize to the case of non-symmetric maps.

Basic examples of unimodal maps are given by the unimodal polynomial maps

$$p_\tau : I \to I, \quad p_\tau(x) = \tau - 1 - \tau x^d,$$

where d is even and $\tau \in [1/d, 2]$ is a real parameter.

Suppose that q is a periodic point of period n. We will let $\overline{q} = \{f^k(q)\}_{k=0}^{n-1}$ denote its cycle. Let $\lambda = (Df^n)(q)$ be its *multiplier*. The cycle $\overline{q}$ is called *repelling* if $|\lambda| > 1$, *parabolic* if $|\lambda| = 1$, *attracting* if $|\lambda| < 1$, and *superattracting* if $\lambda = 0$.

The *basin of attraction* $D(\overline{q})$ of an attracting cycle $\overline{q}$ is $\{x \in I : f^n(x) \to \overline{q}\}$.

A map $f \in \mathcal{U}^3$ is called *Kupka–Smale* if all of its periodic orbits are hyperbolic, and f is called *hyperbolic* if it is Kupka–Smale and the critical point is attracted to a periodic attractor. A hyperbolic map is called *regular* if its critical orbit is neither periodic nor preperiodic. It is well known that regular maps are structurally stable.

2.7.1. *Real puzzle.* Suppose that $f \in \mathcal{U}^3$ is a Kupka–Smale map whose critical point is recurrent but not periodic. Then the first return map of f to its smallest restrictive interval has an orientation reversing fixed point we will call α. An interval J is called *nice* (in the sense of Martens) if the orbits of its boundary points do not intersect $\operatorname{int}(J)$.

The *real Yoccoz puzzle* $\mathcal{P}^{\mathbb{R}}$ for f is a collection of closed intervals P_i^n, $n \in \mathbb{N} \cup \{0\}$, called *real Yoccoz puzzle pieces* such that $P_0^0 = [-\alpha, \alpha]$ and the P_i^n for $n > 0$, are the components of $f^{-n}P_0^0$. Any P_i^n is called a puzzle piece of level n. Intervals of the Yoccoz puzzle containing the critical point are called *critical* and are labelled as P_0^n; it will be convenient to label the puzzle piece of level n that contains the critical value by P_1^n. Aside from these restrictions the indices within a given level are arbitrary. Every Yoccoz puzzle piece is nice. Moreover:

- any non-critical Yoccoz puzzle piece P_i^n is diffeomorphically mapped onto some other puzzle piece $P_{k(i)}^{n-1}$;
- any critical Yoccoz puzzle piece P_0^n, $n > 0$, is folded into the Yoccoz puzzle piece P_1^{n-1} containing the critical value $f(0)$ so that $f(\partial P_0^n) \subset \partial P_1^{n-1}$.

Now take a critical Yoccoz puzzle piece $J_0 \in \mathcal{P}^{\mathbb{R}}$ and consider the first landing map L to it; this map is called the *real puzzle map* associated to J_0. The domain of this map consists of a family $\mathcal{J}$ of disjoint Yoccoz puzzle pieces $J_i \in \mathcal{P}^{\mathbb{R}}$, $i \in \mathbb{N}$, satisfying:

- any J_i, $i > 0$, is diffeomorphically mapped by f onto some other interval $J_{k(i)} \in \mathcal{J}$;
- there exists $n_i \in \mathbb{N}$ such that the branch $L|J_i = f^{n_i}|J_i$ diffeomorphically maps J_i onto J_0.

If J and J' are critical puzzle pieces with the properties that J' is a pullback of J by f^n for some n and the map $f^n : J' \to J$ has degree d (in other words, the map $f^{n-1} : \mathrm{Comp}_{f(J')} f^{-(n-1)}(J) \to J$ is a diffeomorphism), we say that J' is a child of J. We say that a map f is *persistently recurrent* if each critical puzzle piece has at most finitely many children and *reluctantly recurrent* otherwise.

The following nested sequence of puzzle pieces about the critical point will play an important role in what follows. Let $I^0 = [-\alpha, \alpha]$ and, given I^n, let $I^{n+1} = \mathcal{L}_0(I^n)$. The sequence $I^0 \supset I^1 \supset I^2 \supset \cdots$ is called the *principal nest*. We say that f has a *central return* at level n if $R_{I^{n-1}}(0) \in I^n$, and that the return is *non-central* otherwise.

2.7.2. *Renormalization of unimodal maps.* A unimodal map f is called renormalizable if there exists an interval J containing the critical point and an integer $n \geq 2$ such that $f^n(J) \subset J$ and the intervals $J, f(J), f^2(J), \ldots, f^{n-1}(J)$ have pairwise disjoint interiors. The smallest such n is called the renormalization period.

Let n be the renormalization period and $J \ni 0$ be the maximal periodic interval of period n as above. This interval is bounded by a periodic point p and its symmetric point. The restriction $f^n|J$ is called the prerenormalization of f.

Let $A : I \to J$ be the affine rescaling mapping -1 to p. Then the map

$$R(f) \equiv A^{-1} \circ f^n \circ A : I \to I$$

is a *renormalization* of f. If a map f is renormalizable, we can repeat the construction of the Yoccoz puzzle for $R(f)$. It is a well-known fact that if a map is infinitely renormalizable, then it is persistently recurrent.

There are two types of renormalizable maps. If f is a renormalizable map with a prerenormalization $Rf \equiv f^m|J : J \to f^m(J)$, then the renormalization is a *satellite renormalization* if there exist i, j, $0 \leq i < j \leq m-1$, such that $\overline{f^i(\mathrm{Dom}(Rf))} \cap \overline{f^j(\mathrm{Dom}(Rf))} \neq \emptyset$. On the other hand, if the sets $\overline{f^i(\mathrm{Dom}(Rf))}$, $0 \leq i < m-1$, are pairwise disjoint, then the renormalization is said to be a *primitive renormalization*.

A unimodal map will be called *Yoccoz* if it is not infinitely renormalizable and has all periodic orbits repelling. A Yoccoz map is called *Misiurewicz* if its critical point is not recurrent.

Suppose that f is a non-renormalizable Yoccoz map. We say that f has *sufficiently big geometry* if there a exists critical puzzle piece J such that $|J|/|\mathcal{L}_0(J)| > C$, where C is a constant to be determined. We say that f has *bounded geometry* if there exists a constant M such that, for any critical puzzle piece J, $|J|/|\mathcal{L}_0(J)| < M$.

2.7.3. *A priori bounds.* In the study of maps with a recurrent critical point, many nice properties follow from there being definite space between nested puzzle pieces.

PROPOSITION 2.1. [**18**, Lemmas 3.7 and 3.8] *Suppose that f is a reluctantly recurrent Yoccoz map. Then for any $C > 0$, there exists an n such that*

$$|I^n|/|I^{n+1}| > C.$$

For general non-renormalizable Yoccoz maps, we have the following proposition.

PROPOSITION 2.2. [**30**, Theorem A] *There exists $\delta > 0$ with the following property. Suppose that f is a non-renormalizable Yoccoz map with recurrent critical point. Let $\{n_k\}$ denote the sequence of levels of the principal nest where f has a non-central return. Then*

$$|I^{n_k}|/|I^{n_k+1}| > 1 + \delta.$$

Let us remark that this is in stark contrast to the quadratic case where much more is true. The following result is due to Lyubich [**18**], and holds in general by work of Kozlovski [**11**].

PROPOSITION 2.3. *Suppose that f is a non-renormalizable unimodal map with recurrent, non-degenerate, critical point. Let $\{n_k\}$ denote the sequence of levels of the principal nest where f has a non-central return. Then*

$$|I^{n_k}|/|I^{n_k+1}| > C\Lambda^k$$

for some constants $C > 0$ and $\Lambda > 1$.

3. *Complex bounds*

3.1. *Dynamics of complex return maps.* Let W be a quasidisc and let $\{W_j\}$ be a family of at least two quasidiscs inside W with pairwise disjoint closures such that $0 \in W_0$. Assume additionally that

$$\inf\ \mathrm{mod}(W \setminus \overline{W_j}) > 0,$$

and, in case there are infinitely many W_j, that $\mathrm{diam}(W_j) \to 0$.

An *R-map* (or complex return map) is a holomorphic map $f : \bigcup W_j \to W$ such that for any $j \neq 0$, $f|W_j$ is a univalent map onto W, and $f|W_0$ is a d-to-1 covering of W branched at 0.

If $f : \bigcup W_i \to W$ is a complex return map and $n \in \mathbb{N} \cup \{0\}$, we let $Z^n = f^{-n}(W)$. The connected components of Z^n are called the puzzle pieces of level n. For $x \in Z^n$, we let $Y^n(x) = \mathrm{Comp}_x Z^n$. If Y is a puzzle piece containing 0, we say that Y is *critical* and we denote the critical puzzle piece of level n by Y^n. We say that f has *bounded geometry* if there exists a constant C such that

$$\mathrm{mod}(Y^n \setminus \mathcal{L}_0(Y^n)) \leq C$$

for all critical puzzle pieces Y^n.

We define the *filled Julia set* $K(f)$ as $\bigcap_n Z^n$.

An R-map f is called *renormalizable* if there exist a puzzle piece $V = Y^n(0)$ and $p \in \mathbb{N}$ such that $V \subset f^p(V)$, the map $f^p : V \to f^p(V)$ is a degree d map, the puzzle pieces $f^j(V)$, $1 \leq j \leq p$, are pairwise disjoint, and $f^{mp}(0) \in V,\ m > 0$. The map $R(f) = f^p|V$ with minimal n as above is called the renormalization of f. It is a unicritical polynomial-like map with connected Julia set.

We say that f has *well-defined combinatorics up to level n* if 0 belongs to the interior of a puzzle piece of level n. If f has well-defined combinatorics at all levels, we say that f is *combinatorially recurrent* if the orbit of the critical point visits all critical puzzle pieces.

A critical puzzle piece Y^n is a called a *child* of a critical puzzle piece Y^m where $n > m$ if the map $f^{n-m} : Y^n \to Y^m$ is unicritical with a degree d critical point. If f is combinatorially recurrent, then every critical puzzle piece has a child. The *first child* of a puzzle piece Y^n coincides with the critical component of the domain of the first return map to Y^n.

Suppose that $f : U \to V$ is a non-renormalizable R-map with recurrent critical point. Then we can construct the principal nest for f, $V \equiv V^0 \supset V^1 \supset V^2 \supset \cdots$, where V^i is the first child of V^{i-1}. With this terminology, the definitions of persistently recurrent, reluctantly recurrent, central return and non-central return for puzzle maps are the same as for unimodal maps.

Let $f : \bigcup W_j \to W$ and $\tilde{f} : \bigcup \tilde{W}_j \to W$ be two R-maps, and let h be a homeomorphism of $\mathbb{C}$ equivariant on $\bigcup \partial W_j$. If $h(f(0)) = \tilde{f}(0)$, then for each j there is a homeomorphism $\psi_j : \mathrm{cl}\ W_j \to \mathrm{cl}\ \tilde{W}_j$ coinciding with h on ∂W_j and such that $h \circ f = \tilde{f} \circ \psi_j$ on W_j. Let

$$h_1 = \begin{cases} \psi_j & \text{on } W_j, \\ h & \text{on } \mathbb{C} \backslash \bigcup W_j. \end{cases}$$

Since diam $W_j \to 0$, h_1 is a homeomorphism of $\mathbb{C}$. It is called the lift of h.

We say that a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ is a *combinatorial equivalence* between f and $\tilde{f}$ if it is equivariant on $\bigcup \partial W_j$ and the lift h_1 of h is homotopic to h relative to $\bigcup \partial W_j \cup \mathrm{orb}_f(0)$.

We say that two R-maps are *topologically equivalent* (*qc-equivalent*), if there exists a continuous (qc) map ϕ such that $f \circ \phi = \phi \circ g$. If f and g are qc conjugate by a map ϕ and additionally $\overline{\partial}\phi$ vanishes on $K(f)$, then f and g are said to be *hybrid equivalent*.

3.1.1. *Geometry of complex puzzle maps.* Suppose that $f : \bigcup U_j \to V$ is a non-renormalizable R-map whose critical point is recurrent, and let

$$V^0 \supset V^1 \supset V^2 \supset \cdots$$

be its principal nest.

PROPOSITION 3.1. [**3**, Theorem 4.5] *Suppose that* $\{n_k\}$ *is the sequence of levels such that the return map* $f_{n_k} : V^{n_k} \to V^{n_k - 1}$ *is non-central. There exists* $\delta > 0$ *such that* $\mathrm{mod}(V^{n_k} \backslash V^{n_k + 1}) > \delta$.

In the quadratic case, by [**19**], it is known that this quantity grows at least linearly.

Let V be the range of an R-map and $\rho > 0$. We say that V is *ρ-nice* if for any $x \in V \cap \omega(c)$ we have that $\mathcal{L}_x(V) \subset V$ and $\mathrm{mod}(V \backslash \mathcal{L}_x(V)) \geq \rho$. We say that V is *ρ-fat* if there exist puzzle pieces P^+, P^- with $P^+ \supset V \supset P^-$ such that $(P^+ \backslash P^-) \cap \omega(c) = \emptyset$, $\mathrm{mod}(P^+ \backslash V) \geq \rho$ and $\mathrm{mod}(V \backslash P^-) \geq \rho$.

If $\rho > 1$, we say that a simply connected domain U has *ρ-bounded geometry with respect to $x \in U$* if there exist r, R with $0 < r \leq R$ such that $B(x, r) \subset U \subset B(x, R)$ and $R/r < \rho$, where $B(x, r)$ denotes the disc of radius r centred at x. A domain U is said to have *ρ-bounded geometry* if there is an $x \in U$ such that U has ρ-bounded geometry with respect to x.

We will need to make use of the added geometric control of the nest of puzzle pieces called the *enhanced nest* or *KSS nest*. We will denote it by $E^0 \supset E^1 \supset E^2 \supset \cdots$. Let us describe it briefly in the unicritical case. Suppose that f is a non-renormalizable R-map with persistently recurrent critical point. If P is any critical puzzle piece, we let $\Gamma(P)$ denote the smallest child of P, and let ν be the positive integer such that $f^\nu(\Gamma(P)) = P$. Then by [**13**, Lemma 8.1], $f^\nu(c) \in \mathcal{L}_0(P)$ and $\omega(0) \cap \Gamma(P) \subset \mathrm{Comp}_0(f^{-\nu}(\mathcal{L}_0(P)))$. We now set $\mathcal{A}(P) = \mathrm{Comp}_0(f^{-\nu}(\mathcal{L}_0(P)))$. Notice that in the unicritical setting the operator Γ has the properties required of the operator $\mathcal{B}$ defined in [**13**] (see Lemmas 8.1 and 8.2 of that paper). Now, let E^0 be any critical puzzle piece. If E^i is defined, we set $E^{i+1} = \Gamma^6(\mathcal{A}(E^i))$.

PROPOSITION 3.2. ([**14**, Theorem 7.1], see also [**13**]) *Suppose that f is a non-renormalizable R-map with persistently recurrent post-critical set. There exist $D \geq 2$, $\delta > 0$ and $\delta' > 1$ such that the sequence of critical puzzle pieces,*

$$E^0 \supset E^1 \supset E^2 \supset \cdots ,$$

defined above satisfies the following properties:

- *E^{i+1} is a pullback of E^i of degree $\leq D$;*
- *E^i is δ-nice, δ-fat and has δ'-bounded geometry with respect to the critical point.*

We let $E^i_{\mathbb{R}} = E^i \cap \mathbb{R}$, and call the sequence of puzzle pieces $E^0_{\mathbb{R}} \supset E^1_{\mathbb{R}} \supset E^2_{\mathbb{R}} \supset \cdots$ the *real enhanced nest* for f.

3.1.2. *Generalized polynomial-like maps*

Definition 3.1. Let V be a simply connected domain and let $\{U_0, U_1, \ldots, U_n\}$ be a collection of simply connected, pairwise disjoint domains that are compactly contained in V. Set $U = U_0 \cup U_1 \cup \cdots \cup U_n$. A *generalized polynomial-like map* $f : U \to V$ is a holomorphic map such that $f|_{U_i} : U_i \to V$ is a d_i-to-1 covering for each i.

A generalized polynomial-like map whose domain consists of a single component is called a *polynomial-like* map.

Throughout, we will restrict to the case of a map

$$f : \bigcup_{i=0}^{n} U_i \to V$$

such that $f|U_0$ is a d-to-1 covering of V and, for $i \neq 0$, $f|U_i$ maps U_i univalently onto V.

Remark 3.1. Generalized polynomial-like maps possessing at least two connected components in their domains are R-maps, so that the results stated for R-maps hold for these maps as well.

3.2. *Complex puzzle maps.* Let $U \subset \mathbb{C}$ be a bounded open set. We say that a holomorphic function $f : U \to \mathbb{C}$ belongs to the class $A^1(U)$ if f and its derivative f' admit continuous extensions to the closure $\overline{U}$. We will use the same notation, f and f', for the extensions. We endow $A^1(U)$ with the seminorm

$$\|f\|_1 = \max_{z \in \overline{U}} |f'(z)|. \tag{3.1}$$

If $f \in A^1(U)$, $f|\overline{U}$ is a homeomorphism onto its image and f' does not vanish on $\overline{U}$, we say that $f|\overline{U}$ is a *diffeomorphism onto its image*. If U is a bounded connected open set then $\|\cdot\|_1$ is a Banach norm in the subspace of functions vanishing at a given point $z \in \overline{U}$.

Definition 3.2. A map $f \in A^1(U)$ is called a *puzzle map* if:

- $U = \bigcup U_i$ is a countable union of quasidiscs U_i, $i \geq 0$, called puzzle pieces, with pairwise disjoint closures, and $U_0 \ni 0$;
- for $i > 0$, f is a diffeomorphism of $\overline{U_i}$ onto some $\overline{U_j}$;
- there exists a sequence n_i such that $f^{n_i}|U_i$ is a diffeomorphism onto U_0;
- 0 is a critical point of f and f' does not vanish on ∂U_0;
- $f|U_0$ is a degree d covering map onto its image.

Given a puzzle map, we can easily construct a complex return map to U_0 whose domain is given by the collection $\mathcal{L}_x(U_0)$ where $x \in U_0$ has a forward iterate that lands in U_0.

3.3. *Complex bounds for real maps*

PROPOSITION 3.3. ([**27**, Theorem 3]; [**2**, Lemma 5.5]) *Suppose that f is a non-renormalizable Yoccoz map with minimal post-critical set. Then there exists an arbitrarily small critical puzzle piece A such that the following holds. Suppose that $\{A_i\}$ is the domain of the first return map associated to A. Then there exists a collection of open, real-symmetric quasidiscs $\{U_i\}$ in $\mathbb{C}$ with $U_i \cap \mathbb{R} = A_i$, such that $f : \bigcup U_i \to \mathbb{C}$ is a puzzle map.*

We obtain the nth *generalized renormalization* of a puzzle map by restricting the domain of the return map to V^n to the components of the domain that intersect the post-critical set. The following corollary is immediate.

COROLLARY 3.1. *Suppose that f is a real analytic unimodal map that is at most finitely renormalizable with minimal post-critical set and recurrent critical point. Then f possesses a generalized polynomial-like generalized renormalization.*

Given $0 < \theta \leq \pi/2$, and $A = [a, b] \subset \mathbb{R}$, we define the *Poincaré disc with angle θ based on the interval A*, denoted by $D_\theta(A)$, as the intersection of D_1 and D_2 where $D_1 \cap \mathbb{R} = A$ and the boundary of D_1 intersects the real line with angle θ and D_2 is the image of D_1 under reflection about the real line.

Let us fix a deep level n of the principal nest for f and let $A_0 = I^n$. Then A_0 is a nice interval for f and we may associate with it the real puzzle $\{A_j\}$.

PROPOSITION 3.4. [**2**, Lemma 5.5] *Let $0 < \phi < \psi < \gamma < \pi/2$ and $k > 0$. There exists a constant $C > 0$ such that the following holds. Suppose that f is a non-renormalizable Yoccoz map. If I^{n-1} is a sufficiently deep level of the principal nest and $|I^{n-1}|/|I^n| > C$, then there exists a sequence of open Jordan discs, $\{U_j\}$, such that $D_\phi(A_j) \subset U_j \subset D_\psi(A_j)$ and $U_0 = D_{(\phi+\psi)/2}(A_0)$ with the following properties:*

(1) *if $j \neq 0$ and $f(A_j) \subset A_k$, then $f : U_j \to U_k$ is a diffeomorphism;*

(2) *if $f(A_0) \cap A_j \neq \emptyset$, then $\mathrm{mod}(f(U_0)\backslash\overline{D_\gamma(A_j)}) > k$.*

A map f with an associated puzzle as in this lemma will be called a *geometric puzzle map*. Because of the decay of geometry property of quadratic maps, the above proposition holds for all non-renormalizable quadratic maps with recurrent critical point, and the more general Proposition 3.3 is not necessary.

For infinitely renormalizable maps, the connection between real analytic unimodal maps and polynomial-like maps comes from the fact that sufficiently high renormalizations of unimodal real analytic maps are unicritical polynomial-like maps.

THEOREM 3.1. ([**15**, see §11, Theorem A]) *Let f be a real analytic unimodal infinitely renormalizable map and $f^s : J \to J$ be an arbitrary renormalization of this map to some sufficiently small periodic interval $J \subset \mathbb{R}$ containing c. Then this map has a polynomial-like extension $U \to U'$ with:*

- *the modulus of $U'\backslash U$ bounded from below by δ;*
- *the diameter of U' at most C times the diameter of J.*

Here δ depends only on the degree of the map, d, and C is universal. In fact, δ is asymptotically like const/d as $d \to \infty$. Moreover, if f is not $s/2$ renormalizable, then $U' \cap \omega(c) \subset J$.

Note that this property is robust; that is, if it holds for some map $f_0 \in \mathcal{U}$ then it holds with the same n for nearby maps.

3.4. *Holomorphic motions of puzzle maps.* Let f be a Kupka–Smale map. Let A denote the set of attracting periodic orbits of f and let B denote the basins of the attracting periodic orbits. We say that a homeomorphism $h : I \to \mathbb{C}$ is *f-admissible* if the following holds. Let T be a periodic component of $B\backslash A$ that does not contain 0, and writing $T = (a, b)$ with $|a| < |b|$, we have that the interval $[-a, a]$ is nice. Then h takes $d = (a + b)/2$ to $h(d) = (h(a) + h(b))/2$ and $h|[d, f^q(d)]$ is affine, where q is the period of T.

Suppose that $f : \bigcup U_j \to U$ is a puzzle map. Let $U_1 = [l, r]$ be the component of the domain of the return map that contains the critical value. Let V be the union of all U_j such that $U_j \cap \mathbb{R} \subset [-1, r]$. Let $\mathcal{V} \subset \mathcal{A}_a$ be a real-symmetric neighbourhood of f. We will say that the puzzle *persists* in $\mathcal{V}$ if there exists a real-symmetric holomorphic motion h over $\mathcal{V}$ given by a family of transition maps $h_g : \mathbb{C} \to \mathbb{C}$, $g \in \mathcal{V}$, such that:

(1) $h_g|C\backslash\Omega_a = \mathrm{id}$;

(2) $g \circ h_g|V\backslash V^0 = h_g \circ f$, $g \circ h_g|\partial V^0 = f$;

(3) $h_g|I$ is f-admissible and $g \circ h_g|([-1, r]\backslash V) = h_g \circ f$.

The last condition in this definition defines h_g uniquely in $[-1, r]\backslash\overline{V}$.

For maps with sufficiently big geometry we have the following proposition.

PROPOSITION 3.5. ([**2**, Lemma 5.6]; [**6**, Lemma A.2]) *Let* $0 < \phi < \psi < \gamma < \pi/2$ *and let* $k > 0$. *There exists a constant* $C > 0$ *with the following property. If* $f \in \mathcal{U}_a$ *is a non-renormalizable Yoccoz map with the property that there exists a sufficiently small critical puzzle piece* Y *such that* $|Y|/|\mathcal{L}_0(Y)| > C$, *then there exists a geometric puzzle for* f *with parameters* (ϕ, ψ, γ, k) *that persists on a neighbourhood of* $\mathcal{V} \subset \mathcal{A}_a$ *of* f.

For maps with minimal post-critical set we have a similar result.

PROPOSITION 3.6. *Suppose that* f *is a non-renormalizable Yoccoz map with minimal post-critical set. There exist arbitrarily small real critical puzzle pieces* J *such that the real puzzle map associated to* J *extends to a puzzle map that persists over a neighbourhood* $\mathcal{V} \subset \mathcal{A}_a$ *of* f.

Proof. The proof of this result is adaptation of part of the proof of Lemma 12.5 of [**17**]. We will only give an outline.

First, the Cantor set Q of points that never enter J_0 is contained in a persistent Markov family $\{M_j\}$. By [**2**, Proposition 2.11], there is a holomorphic motion $h_g : \mathbb{C} \to \mathbb{C}$ over a neighbourhood of f such that $h_g \circ f = g \circ h_g \circ g$ on $\bigcup M_j$. All but finitely many U_i are compactly contained in $\bigcup M_j$. Let $\mathcal{I}$ be the set of all U_i that are not compactly contained in $\bigcup M_j$, and let $\mathcal{J}$ be the set of all U_i that are compactly contained in $\bigcup M_j$, but with $f(U_j) \in \mathcal{I}$; $\mathcal{J}$ is also a finite set. Shrinking the neighbourhood $\mathcal{V}$, if needed, we may suppose that, for each $g \in \mathcal{V}$, h^i_g is defined for $U_i \in \mathcal{I} \cup \mathcal{J}$. If $U_i \notin \mathcal{I} \cup \mathcal{J}$, then there is a unique k such that $f^k(U_i) = U_j \in \mathcal{J}$. We then define $h^i_g|U_i$ so that $g^k \circ h^i_g = h^j_g \circ f^k$. We extend each of the holomorphic motions h^i_g to a normalized holomorphic motion of $\mathbb{C}$. We need to show that we can restrict the domains of these holomorphic motions to an open set $\mathcal{V}$ about f, so that the puzzle pieces do not collide as g varies in $\mathcal{V}$.

There exists a constant $r > 0$ such that if $\operatorname{diam} U_j < r$, then $U_j \subset \bigcup M_j$. Such components are separated by the Markov partition. Thus, if U_i and U_j are two such components, $h^i_g(U_i)$ will be disjoint from $h^j_g(U_j)$ for $g \in \mathcal{V}$. There are only finitely many components U_j with diameter greater than r, so we can restrict the domains of the holomorphic motions so that they do not collide.

Finally, we need to show that we can restrict the domains of the holomorphic motions so that if U_i, U_j are components of the domain of the return map with $\operatorname{diam} U_i > r$ and $\operatorname{diam} U_j < r$, then for $g \in \mathcal{V}$, $h^i_g(U_i)$ and $h^j_g(U_j)$ are disjoint. We argue by way of contradiction. Suppose that U_i and U_j are two such components that $h^i_g(U_i) \cap h^j_g(U_j)$ intersect. Let n_i be such that $f^{n_i} : U_i \to U$. Then n_i is bounded from above by a universal constant N. Hence $g^k(U_j)$ is contained in a small neighbourhood of the real line for k, $0 \le k \le N$. Now we have that there exists a constant M such that $f^{n_i+M}(U_j)$ lies in a small neighbourhood of the α fixed point of f. Since $f^{n_i+M}(U_i)$ and $f^{n_i+M}(U_j)$ intersect, it follows that iterates of g repel points to some definite distance h from the real axis before they return to U. This cannot happen because the map is expanding away from the critical point.

The same argument applies to components of the domain and $f(U_0)$. □

3.5. *Rigidity and Sullivan's pullback argument.* For the results below, the proofs of [**2**] need to be modified slightly because it is not known if in degrees greater than two the filled Julia sets of non-renormalizable Yoccoz maps have measure zero; however, the proposition below allows the proofs to go through. We will not repeat them here.

PROPOSITION 3.7. ([**16**] and [**2**, §A.6]) *Suppose that f is an $\mathbb{R}$-symmetric R-map, or a real polynomial-like map with all cycles repelling. Then f carries no invariant line field on its Julia set.*

This is Sullivan's pullback argument, but for puzzle maps.

PROPOSITION 3.8. [**2**, Theorem 6.2] *Let us consider two puzzle maps f and $\tilde{f}$ with all periodic orbits hyperbolic. Let U_f denote the puzzle for f. Let h be a qc combinatorial equivalence between f and $\tilde{f}$. Then there is a qc homeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $H \circ f = \tilde{f} \circ H$ on U_f, $H = h$ on $\mathbb{C}\backslash U_f$, and $\mathrm{Dil}(H) \le \mathrm{Dil}(h)$. If there are no invariant line fields on $K(f)$ or if f and $\tilde{f}$ are hyperbolic maps in the same hybrid class, then $\mathrm{Dil}(H) \le \mathrm{Dil}(h|\mathbb{C}\backslash U_f)$.*

One consequence of the pullback argument is that conjugacies between preperiodic or hyperbolic maps close to Yoccoz maps respect the puzzle structure. Let $\mathcal{C}_g$ denote the topological conjugacy class of a map g.

LEMMA 3.1. [**2**, Lemma 6.3] *There exists a constant $L > 0$ with the following property. Let $f \in \mathcal{U}_a$ be a Yoccoz map, and let U_f be a puzzle which persists in an ϵ-neighbourhood of $f \in \mathcal{A}$. Let $\mathcal{V}$ be an $\epsilon/2$-neighbourhood of f. If $g \in \mathcal{U}_a \cap \mathcal{V}$ is preperiodic or hyperbolic and $\tilde{g}$ belongs to the same connected component of $\mathcal{C}_g \cap \mathcal{V}$, then there is a normalized L-qc homeomorphism $h : \mathbb{C} \to \mathbb{C}$ equivariant with respect to g and $\tilde{g}$ on U_g, where U_g denotes the puzzle for g.*

THEOREM 3.2. *Suppose that f and $\tilde{f}$ are real analytic unimodal maps with all periodic cycles repelling. Then if f and $\tilde{f}$ are topologically conjugate, they are qs-conjugate.*

Proof. For infinitely renormalizable maps this follows from [**15**, Theorem A] and the rigidity theorem of [**13**]. For finitely renormalizable maps we may repeat the proof of Theorem B$''$ of [**17**] (where the same result is proved for a class of maps that contains covering maps of the circle). For maps with a non-recurrent critical point one may refer to [**24**] for a proof. □

By standard considerations (for instance, see [**12**]), this implies the following corollary.

COROLLARY 3.2. *Suppose that $\{f_\lambda\}$ is an analytic family of unimodal maps that does not possess a subfamily contained in a single topological conjugacy class. Then the set of hyperbolic maps in the family is dense in $\{f_\lambda\}$.*

4. *Infinitesimal theory and the hybrid lamination*

4.1. *A variational formula.* Let S_n denote the iteration operator $f \mapsto f^n$ as a map acting between spaces of analytic functions. Linearizing $(f + \varepsilon v)^n$, we obtain by induction the following formula for the differential of S_n:

$$v^n \equiv DS_n(f)v = Df^{n-1} \circ f \sum_{k=0}^{n-1} \frac{v \circ f^k}{Df^k \circ f} = Df^n \sum_{k=0}^{n-1} (f^k)^* \left(\frac{v}{f'}\right). \tag{4.1}$$

Note that if f_t is a one-parameter family of analytic maps such that

$$\frac{d}{dt} f_t \bigg|_{t=0} = v,$$

then

$$\frac{d}{dt} f_t^n \bigg|_{t=0} = v^n.$$

Applying (2.1) to the iterates of f^* and summing, we see that if α is equivariant with respect to (f, v) on $\bigcup_{k=0}^{n-1} f^k(X)$, then it is equivariant with respect to (f^n, v^n) on X:

$$(f^n)^*\alpha - \alpha = \sum_{k=0}^{n-1} (f^k)^*(f^*\alpha - \alpha) = \sum_{k=0}^{n-1} (f^k)^* \left(\frac{v}{f'}\right) = \frac{v^n}{Df^n}.$$

Rewriting this equation, we have

$$\alpha = (f^n)^*\alpha - \frac{v^n}{Df^n}.$$

Finally, note that if β is obtained by n consecutive lifts of α by (f, v), then β is the lift of α by (f^n, v^n).

4.2. *The infinitesimal pullback argument and the key estimate.* Let T_f be the space of all holomorphic vector fields v with the property that there exists a qc vector field α on $\mathbb{C}$ satisfying

$$v = \alpha \circ f - f'\alpha$$

on orb(0). The space T_f allows us to study complex perturbations of vector fields.

Given a puzzle map, f, the infinitesimal pullback argument allows us to extend qc vector fields that are defined and equivariant on certain parts of the domain of f, for example on the post-critical set of f or on the boundary of the domain of f, to the entire domain.

The next lemma provides one step of the infinitesimal pullback argument.

LEMMA 4.1. *Let $\Omega \ni 0$ be a quasidisc. Consider a map $f \in A^1(\Omega)$ whose derivative does not vanish on $\overline{\Omega}\backslash\{0\}$. Assume that $f : \overline{\Omega} \to f(\overline{\Omega})$ is either a diffeomorphism or a degree d branched covering ramified at 0. Let $v \in \mathcal{B}(U)$. Let α and β be qc vector fields on $\mathbb{C}$ such that $\beta|\partial\Omega$ is the lift of α by (f, v). Moreover, if f is a degree d branched covering, we assume that $v(0) = \alpha(f(0))$, and $v^{(i)}(0) = 0$ for $1 \leq i \leq d-2$ (notice that this restriction is irrelevant in the case when $d = 2$). Then there exists a qc vector field γ such that $\gamma|\Omega$ is the lift of α by (f, v), $\gamma|\mathbb{C}\backslash\Omega = \beta$, and*

$$\|\overline{\partial}\gamma\|_\infty \leq \max\{\|\overline{\partial}\alpha\|_\infty, \|\overline{\partial}\beta\|_\infty\}.$$

Proof. Define a continuous vector field γ on $\mathbb{C}\backslash\{0\}$ by letting $\gamma = \beta$ on $\mathbb{C}\backslash\Omega$ and letting $\gamma = (\alpha \circ f - v)/f'$ on $\Omega\backslash\{0\}$. If f is a diffeomorphism then γ clearly extends to 0.

Now, assume that f is a degree d branched covering. Since the modulus of continuity of qc vector fields is $\phi(x) = -x\ \ln(x)$, for z near 0 we have:

$$|\alpha(f(z)) - \alpha(f(0))| = O(\phi(|f(z) - f(0)|)) = O(\phi(|z|^d)) = O(-d|z|^d \ln(|z|)).$$

Since $v(0) = \alpha(f(0))$ and f' has a zero of degree $d-1$ at 0, we have that, near 0,

$$\begin{aligned}\gamma(z) &= \frac{\alpha \circ f(z) - v(z)}{f'(z)} \\ &= \frac{\alpha \circ f(z) - \alpha \circ f(0) + \alpha \circ f(0) - v(z)}{f'(z)} \\ &= \frac{v(0) - v(z)}{f'(z)} + \frac{\alpha \circ f(z) - \alpha \circ f(0)}{f'(z)} \\ &= \frac{v(0) - v(z)}{f'(z)} + O\left(\frac{-z^d \ln(z)}{z^{d-1}}\right).\end{aligned}$$

Hence,

$$\gamma(z) = \frac{v(0) - v(z)}{f'(z)} + O(\phi(|z|)).$$

Thus γ has a continuous extension to 0 since $v^{(i)}(0)$ vanishes for $1 \leq i \leq d-2$.

γ is quasiconformal on $\mathbb{C} \backslash (\partial\Omega \cup \{0\})$ (because α is qc), and since quasiarcs and isolated points are qc removable, γ is quasiconformal on all of the complex plane.

Since lifts preserve the norm of qc vector fields (equation (2.2)), we have that $\|\bar{\partial}\gamma\|_\infty = \|\bar{\partial}\alpha\|_\infty$ on Ω, while $\|\bar{\partial}\gamma\|_\infty = \|\bar{\partial}\beta\|_\infty$ on $\mathbb{C}\backslash\Omega$. Since quasiarcs are removable, the desired estimate follows. □

With this lemma established, we have the following result, which follows exactly as in the quadratic case (see [**2**, Theorem 6.5]).

PROPOSITION 4.1. (Infinitesimal pullback argument) *Let $f : U \to \mathbb{C}$ be a puzzle map whose critical point does not escape $\overline{U}$, and let v be a tangent vector field at f. Assume that there exists a normalized qc vector field β on $\mathbb{C}$ which is equivariant on $\partial U \cup$ orb(0). Then there exists an equivariant on U qc vector field α with $\|\bar{\partial}\alpha\|_\infty \leq \|\bar{\partial}\beta\|_\infty$ which coincides with β on $\mathbb{C}\backslash U$. Furthermore, if there are no invariant line fields on $K(f)$, then $\|\bar{\partial}\alpha\|_\infty \leq \|\bar{\partial}\beta|\mathbb{C}\backslash U\|_\infty$.*

We say that a preperiodic or hyperbolic complex map g has *special combinatorics* with respect to $\mathcal{V}$, a complex neighbourhood of g, if the connected component of $g \in \mathcal{V} \cap \mathcal{C}_g$ contains a real map. We let L_g denote the linear map that associates to any tangent vector $v \in T_g$ the unique qc vector field α on the post-critical set such that $v = \alpha \circ f - f'\alpha$ and $v(0) = \alpha(c_1)$.

THEOREM 4.1. (Key estimate, [**2**]) *Let $f \in \mathcal{U}_a$ be either a Yoccoz map or a hyperbolic map. There exists a neighbourhood $\mathcal{V}$ of f in $\mathcal{A}_a$ and a constant $C > 0$ such that, for any g with special combinatorics with respect to $\mathcal{V}$, the operator norm of L_g is bounded by C.*

Proof. The proof of this theorem is the same as in [**2**, §6]. We will only give an outline. Consider a persistent puzzle for f given by Proposition 3.6. Take an $\varepsilon > 0$ such that this puzzle persists in a ε-neighbourhood of f and let H_g be the associated holomorphic motion. Let $\mathcal{V}$ be an $\varepsilon/2$-neighbourhood of f and let $C = 2/\varepsilon$. Given $g \in \mathcal{V}$ with special combinatorics and $v \in T_g$ with $\|v\|_a = 1$, let $h_\lambda = H_{g+\lambda v} \circ H_g^{-1}$, $\lambda \in \mathbb{D}_{\varepsilon/2}$, where we define $H_{g+\lambda v}$ by $H_g \circ (\mathrm{id} + \lambda v)$. Let

$$\beta_0 = \frac{d}{d\lambda} h_\lambda \Big|_{\lambda=0}.$$

Notice that β_0 is equivariant on ∂U_g with respect to (g, v), where U_g denotes the puzzle for g. By [**8**, Theorem 2], μ_{h_λ}, where $\overline{\partial} h_\lambda = \mu_{h_\lambda} \partial h_\lambda$, is holomorphic on $\mathbb{D}_{\varepsilon/2}$ with values in the unit ball of $L^\infty(\mathbb{C})$. By Lemma 2.5,

$$\overline{\partial} \beta_0 = \frac{d}{d\lambda} \mu_{h_\lambda} \Big|_{\lambda=0}.$$

Since $\mu_{h_0} = 0$, the Schwarz lemma implies $\|\overline{\partial}\beta_0\|_\infty \le C$.

If g is preperiodic, by Proposition 3.7, g has no invariant line fields $K(g)$. By Lemma 3.1, the special combinatorics assumption implies that $g|U_g$ is qc-conjugate to some real map $\tilde{g} \in \mathcal{V}$. Now, Proposition 4.1 implies that there exists a qc vector field α, equivariant on U_g, coinciding with β_0 on $\mathbb{C}\backslash U_g$ such that $\|\overline{\partial}\alpha\| \le C$.

If g is hyperbolic, it is not hard to show that there exists a qc vector field α on $\mathbb{C}$ that is conformal and equivariant on $\overline{\mathrm{orb}(0)}$ (see [**2**, Lemma 6.10]). We can then create a new vector field γ by gluing β_0 on the complement of U_g with α on a neighbourhood of $\overline{\mathrm{orb}(0)}$. Applying Proposition 4.1, we have that there exists a vector field β that is equivariant on U_g, conformal in the basin of attraction of $\mathrm{int}(K(g))$ and satisfies

$$\|\overline{\partial}\beta | \mathbb{C}\backslash K(g)\|_\infty \le C.$$

Since the Julia set of a hyperbolic puzzle map has measure zero, we are done. □

4.3. *Infinitesimal structure in spaces of generalized polynomial-like maps.* Let $U = U_0 \cup U_1 \cup \cdots \cup U_n$ where the U_i are simply connected open subsets of the plane. We will denote by $\mathcal{B}_{\mathrm{nor}}(U) \subset \mathcal{B}(U)$ the closed affine space of functions f such that the domain U contains 0 and -1 and furthermore $f(-1) = -1$ and $f^{(i)}(0) = 0$, for $1 \le i < d$. We can identify the tangent space at f in $\mathcal{B}_{\mathrm{nor}}(U)$ with a family of finitely many vector fields $\{v_j\}$ such that v_j is defined on U_j, the vector field vanishes at -1, and v_0 is normalized at the origin as $c + az^{d+1} + \cdots$.

Definition 4.1. The tangent space to the hybrid class of f in $\mathcal{B}_{\mathrm{nor}}(U)$ is the set of all holomorphic vector fields $v \in T_f\mathcal{B}_{\mathrm{nor}}(U)$ which admit a representation as

$$v = \alpha \circ f - f'\alpha$$

near the filled Julia set, where α is a qc vector field. This set of vector fields will be denoted by $E^h(f)$.

We say that a vector field $v(z)/dz$ is *vertical* if there exists a holomorphic vector field $\alpha(z)/dz$ on $\overline{\mathbb{C}}\backslash K(f)$ vanishing at ∞ such that

$$v(z) = \alpha \circ f(z) - f'(z)\alpha(z)$$

near the Julia set. This set of vector fields will be denoted by $E^v(f)$.

The following proposition follows immediately from density of hyperbolicity and [**29**, Corollary 12.3].

PROPOSITION 4.2. *For any generalized polynomial-like map $f : \bigcup U_i \to V$, the tangent space of f, $T_f\mathcal{B}_{\mathrm{nor}}(U)$, splits into $E^h(f)$, which has codimension one and is the tangent space to $\mathcal{H}_f$, and $E^v(f)$, a one-dimensional subspace transverse to $E^h(f)$.*

4.4. *Properties of the generalized renormalization operator*

LEMMA 4.2. *The generalized renormalization operator is analytic.*

Proof. The generalized renormalization operator is a composition of the iteration operator with the rescaling operator. The rescaling operator is obviously real analytic, and the iterates of f depend analytically on f. □

The following is a trivial consequence of the existence of a persistent puzzle map.

LEMMA 4.3. *Suppose that f is a combinatorially recurrent Yoccoz map with minimal post-critical set, and that $Rf : \bigcup U_i \to U$ is a generalized polynomial-like generalized renormalization of f. Then there exists a neighbourhood $\mathcal{U}$ of f such that any $g \in \mathcal{U}$ is a generalized polynomial-like map defined on $\bigcup h_g(U_i)$, where h_g is a holomorphic motion of the puzzle for f.*

We want to show that $DR(f) : T_f\mathcal{U}_a^{\mathbb{R}} \to T_{Rf}\mathcal{B}_{\mathrm{nor}}(U)$ is *transversally non-singular*; that is, the image of $DR(f)$ contains vectors transverse to $\mathcal{H}^{\mathbb{R}}_{R(f)}$.

LEMMA 4.4. *Suppose that f is a real analytic non-renormalizable unimodal map with minimal post-critical set and recurrent critical point. For any generalized polynomial-like generalized renormalization of f, $R : \mathcal{U}_a \to \mathcal{B}_{\mathrm{nor}}(U)$ ($U = U_0 \cup U_1 \cup \cdots \cup U_n$), the image of $DR(f) : T_f\mathcal{U}_a \to T_{Rf}\mathcal{B}_{\mathrm{nor}}(U)$ is dense.*

Proof. Recall that $S_n : f \mapsto f^n$. The derivative of the generalized renormalization operator is the composition of DS_n with the derivative of a rescaling map. The image of the rescaling map is clearly dense in its range so all that remains to be shown is that the image of S_n is dense onto its range. To see this recall formula (4.1):

$$DS_n v = Df^n \sum_{k=0}^{n-1} (f^k)^* \left(\frac{v}{f'}\right).$$

Let w be a vector field defined in $U_0 \cup U_1 \cup \cdots \cup U_n$. We now define a vector field v so that $DS(f)v = w$. First, set $v = 0$ on

$$\bigcup_{i=0}^{n} \bigcup_{k=1}^{n_i} f^k(U_i).$$

Notice that the sets being considered here are the forward images of the components of the domain of Rf under the original map f, until they land on the range of Rf. When f is non-renormalizable, the closures of these sets are pairwise disjoint.

Now using the formula above with w on the left-hand side, extend v to $\bigcup_{i=0}^{n} U_i$. Notice that all the terms in the sum vanish except the one corresponding to $k = 0$, and we have that

$$v(z) = \frac{w(z)}{Df^{n-1}(f(z))}.$$

v is a vector field vanishing outside U_0 such that $DS_n(f)v = w$. To complete the proof, we use Runge's theorem to approximate v by an even $\mathbb{R}$-symmetric polynomial vector field. □

LEMMA 4.5. *In the setting of the previous lemma, the map*

$$DR(f) : T_f\mathcal{U}_a \to T_{Rf}\mathcal{B}_{\mathrm{nor}}(U)$$

is transversally non-singular.

Proof. Since the image of the derivative of the generalized renormalization operator is dense, its image contains vectors transverse to $\mathcal{H}_f^{\mathbb{R}}$. □

Remark 4.1. In the above argument, we use the fact that the components of the domain of the generalized polynomial-like map are disjoint. In the case of maps that are satellite renormalizable, the small Julia sets intersect, so this property cannot hold. To deal with this case, one may use [**2**, Lemma 4.6], which applies just as well in the higher-degree case.

4.5. *Existence of transverse vector fields and the hybrid lamination.* When f is non-regular, we will show that the leaf of the lamination through f is given by the topological equivalence class of f and we define this to be the *real hybrid class* of f. In the case of hyperbolic maps, it is necessary to refine the topological conjugacy class to obtain codimension-one leaves; this was carried out in [**2, 6**]. With the preparation we have already done, the arguments of [**2, 6**] go through for hyperbolic maps, maps with a non-recurrent critical point and Yoccoz maps with sufficiently big geometry.

PROPOSITION 4.3. ([**2**, Theorem A]; [**6**, Proposition A.1]) *The hybrid class of a hyperbolic map or a map with a non-recurrent critical point is a codimension-one embedded real analytic Banach submanifold in $\mathcal{U}_a$.*

PROPOSITION 4.4. [**2**, Theorem A] *There exists a constant $C_1 > 0$ such that if f is a Yoccoz map with the property that there exists a sufficiently small Yoccoz puzzle piece Y such that $|Y|/|\mathcal{L}_0(Y)| > C_1$, then the hybrid class of f is a codimension-one embedded real analytic Banach submanifold in $\mathcal{U}_a$.*

THEOREM 4.2. *Suppose that $f \in \mathcal{A}_a$ has a minimal post-critical set. Then there exists a neighbourhood $\mathcal{V} \subset \mathcal{A}_a$ of f endowed with a codimension-one holomorphic lamination $\mathcal{L}$ with the following properties:*

(1) *the lamination is real-symmetric;*
(2) *if $g \in \mathcal{V} \cap \mathcal{A}_a^{\mathbb{R}}$ is non-regular, then the intersection of the leaf through g with $\mathcal{A}_a^{\mathbb{R}}$ coincides with the intersection of the topological conjugacy class of g with $\mathcal{V}$;*
(3) *each $g \in \mathcal{V} \cap \mathcal{A}_a^{\mathbb{R}}$ belongs to some leaf of $\mathcal{L}$.*

Proof. By Corollary 3.1 f has a generalized polynomial-like generalized renormalization. By Lemma 4.5 the image of $DR|_f$ contains a vector field transverse to the hybrid class of Rf. Pulling this vector field back by $DR|_f$, we obtain a vector field transverse to the hybrid class of f. The infinitesimal pullback argument implies that this vector field does not lie in T_f. By Corollary 3.2, we can approximate f by a sequence, $\{f_n\}$, of hyperbolic maps. Then by [**2**, Lemma 2.32], since for each f_n, $T_{f_n}\mathcal{H}_{f_n}$ has codimension one, lim sup $T_{f_n}\mathcal{H}_{f_n}$ is either $T_f\mathcal{A}_a$ or it has codimension one in $T_f\mathcal{A}_a$. Moreover, [**2**, Lemma 8.3] implies that lim sup $T_{f_n} \subset T_f$. Hence lim sup $T_{f_n} = T_f$ is a codimension-one subspace, and

$$T_f\mathcal{A}_a = T_f \oplus V_f^{\mathrm{tr}},$$

where T_f is a codimension-one subspace and V_f^{tr} is transverse to the hybrid class of f.

With this proved, Lemma 3.1 and Propositions 3.6 and 4.1 and Theorem 4.1 together imply that the entire discussion of [**2**, §8] goes through for higher-degree maps and we will only provide an outline of it. Since $\mathcal{A}_a$ is an affine space, we can use the decomposition $T_f\mathcal{A}_a = T_f \oplus V_f^{\mathrm{tr}}$ as a coordinate system in the space. For any complex map g close to f we can consider the cone in the tangent space $T_g\mathcal{A}_a$,

$$\mathcal{K}_g^\theta = \{v \in T_g\mathcal{A}_a : \|v^h\| < \tan\theta\|v^{\mathrm{tr}}\|\},$$

where v^h and v^{tr} are the projections of v onto T_f and V_f^{tr} respectively. We claim that the following *cone transversality property* holds. For g sufficiently close to f and θ sufficiently small we have that $\mathcal{K}_g^\theta \cap T_g = \emptyset$. If not, there exist a sequence of maps $g_n \to f$ such that either g_n does not have an invariant line field on $K(g_n)$ or g_n is hyperbolic, and a sequence of vector fields $v_n \in T_{g_n}\mathcal{A}_a$ converging to V_f^{tr}. Let α_n be a qc vector field equivariant with respect to (g_n, v_n) on $\mathrm{orb}_{g_n}(0)$. By the compactness lemma for qc vector fields and Theorem 4.1, $\{\alpha_n\}$ has a subsequence that converges to a qc vector field α that is equivariant on $\mathrm{orb}_f(0)$ with respect to (f, V_f^{tr}). This implies that V_f^{tr} lies in $T_f\mathcal{H}_f$, which is a contradiction [**2**, Lemma 8.1 and Corollary 8.2].

The above discussion implies that if v is a transverse vector field, there exists $\varepsilon > 0$ such that the family $\Sigma_\varepsilon = \{f + tv : t \in (-\varepsilon, \varepsilon)\}$ intersects each hybrid class in at most one point [**2**, Lemma 8.3]. So we can choose a neighbourhood $\mathcal{V} \subset \mathcal{A}_a$ of f such that $\mathcal{K}_g^\theta \cap T_g = \emptyset$ holds and so that $\mathcal{V}$ is a product of Σ_ε with the hybrid classes. We already have that the hybrid classes of hyperbolic maps are analytic codimension-one submanifolds of $\mathcal{A}_a$. The cone transversality property implies that these submanifolds have bounded slope in the coordinate system given by $T_f \oplus V_f^{\mathrm{tr}}$. This together with Lemma 3.1 implies that the hybrid classes of hyperbolic maps are graphs with bounded slope inside of the neighbourhood $\mathcal{V}$ (see [**2**, Lemmas 8.6 and 8.7]). Since the hybrid classes of hyperbolic maps cannot intersect, we have a lamination of $\mathcal{V}$ through a dense subset of Σ_ε. The extension lemma for holomorphic motions promotes it to a lamination through the transversal Σ_ε (compare the proof of Theorem A of [**2**, p. 522]). This completes the proof of this theorem and Theorem A. □

5. *Parameter partition*

5.1. *Families of return and landing maps.* In the discussion below we describe the relationship between the dynamical and parameter planes. For closely related results see [**3**, Lemma 3.4] and [**6**, Lemma 6.12], which have their roots in [**20**].

An *R-family* is a pair $(\mathbf{R}, h)$, where $\mathbf{R}$ is a holomorphic map $\mathbf{R} : \bigcup \mathbf{U}_j \to \mathbf{U}$ such that the fibres $R[\lambda]$ of $\mathbf{R}$ are R-maps, for every j, $\mathbf{R}|\mathbf{U}_j$ is a tube map, and $h = h_{\overline{U[0]}}$ is a holomorphic motion such that $h|(\partial U[0] \cup \bigcup_j \partial U_j[0])$ is special. If additionally $\mathbf{R} \circ \mathbf{0}$ is a diagonal to h we say that $(\mathbf{R}, h)$ is *full*.

We can pass to a family of landing maps from a family of return maps as follows. Suppose that $(\boldsymbol{R}, h)$ is an R-family with fibres $R[\lambda] : U_j[\lambda] \to U[\lambda]$. If $\underline{d} = (d_1, d_2, \ldots, d_m)$ is a sequence of integers, we let $\boldsymbol{U}_{\underline{d}} = \{(\lambda, z) \in \boldsymbol{U} : R^{k-1}[\lambda](z) \in U_{d_k}[\lambda]\}$ and define $\boldsymbol{R}^{\underline{d}} = \boldsymbol{R}^m|\boldsymbol{U}_{\underline{d}}$. We let $\boldsymbol{W}_{\underline{d}} = (\boldsymbol{R}^{\underline{d}})^{-1}\boldsymbol{U}_0$. We define $L(\boldsymbol{R}) : \bigcup \boldsymbol{W}_{\underline{d}} \to \boldsymbol{U}_0$ to be the landing map to $\boldsymbol{U}_0$. To define a holomorphic motion associated to $L(\boldsymbol{R})$, we let the leaf through $z \in \partial U$ be the leaf of h through z. If there is a smallest $U_{\underline{d}}$ such that $z \in U_{\underline{d}}$, we let the leaf through z be the preimage of the leaf of h through $\boldsymbol{R}^{\underline{d}}(z)$ under the map $\boldsymbol{R}^{\underline{d}}$. Finally, we extend $L(h)$ to $\overline{U}$ by the extension lemma.

If $(\boldsymbol{R}, h)$ is a full R-family, $L(h)|(U \cup \bigcup_j \overline{U}_j)$ is special, and we let χ denote the holonomy family of the pair $(L(h)|(U \cup \bigcup_j \overline{U}_j), \boldsymbol{R}(\mathbf{0}))$. We define $\Lambda_{\underline{d}} = \chi(U_{\underline{d}})$ and $\Gamma_{\underline{d}} = \chi(W_{\underline{d}})$.

We will now specialize our discussion to the returns of the critical orbit. Suppose that $(\boldsymbol{R}_0, h_0)$ is a full R-family over Λ^0, a Jordan disc, with the property that the critical value map $\phi : \lambda \mapsto R_0[\lambda](0)$ is a diagonal to h_0. Let χ_0 be the associated holonomy map. Fix $\lambda_0 \in \Lambda^0$, $R_0[\lambda] : \bigcup U_j^0[\lambda] \to U^0[\lambda]$. Suppose that the orbit of the critical point under $R_0[\lambda_0]$ enters $U_0^0[\lambda_0]$. Let $W^0[\lambda_0] = W^0_{\underline{d_0}}[\lambda_0]$ be the domain of $L(R_0[\lambda_0])$ containing $R_0[\lambda_0](0)$. We define $\Lambda^0_{\underline{d}} = \chi_0(U^0_{\underline{d}})$. Set $\Gamma^0 = \Gamma_{\underline{d_0}} = \chi_0(W^0_{\underline{d_0}})$. Define h_1 to be the lift of $L(h_0)$ by $(\boldsymbol{R}_0|U^0, \Gamma^0, W^0)$ wherever it is defined. Let $U^1[\lambda] = U_0^0[\lambda]$. For $z \in U^1[\lambda]$, we define $R_1[\lambda](z)$ to be the return map of z to $U^1[\lambda]$ under $R_0[\lambda]$ for any point z whose orbit returns to $U^1[\lambda]$. The family $(\boldsymbol{R}_1, h_1)$ is full, its fibres are the return maps to $U^1[\lambda]$ of the fibres of $(\boldsymbol{R}_0, h_0)$, and h_1 is a special motion. We let $R_1[\lambda] : \bigcup U_j^1[\lambda] \to U^1[\lambda]$. If the critical orbit under f_{λ_0} enters $U_0^1[\lambda_0]$, then we can repeat the procedure after defining $W^1[\lambda_0] = W^1_{\underline{d_1}}$ to be the component of $L(R_1[\lambda_0])$ that contains the critical value, $\Gamma^1 = \Gamma_{\underline{d_1}}$ and $U^2[\lambda] = U_0^1[\lambda]$. If $R_0[\lambda_0]$ is combinatorially recurrent, we obtain a sequence of R-families $(\boldsymbol{R}_i, h_i)$. We let χ_i be the holonomy family of the pair $(L(h_i)|(U^i \cup \bigcup_j U_j^i), \boldsymbol{R}_i(\mathbf{0}))$, and define $\Lambda^i_{\underline{d}} = \chi_i(U^i_{\underline{d}})$ and $\Lambda^i = \chi_i(U^i[\lambda_0])$. We call the sets $\Lambda^i_{\underline{d}}$ *parapuzzle pieces*. The $\Lambda^i_{\underline{d}}$ are open Jordan discs. We call the nest $\Lambda^0 \supset \Lambda^1 \supset \Lambda^2 \supset \cdots$ the *principal nest of parapuzzle pieces*. We say that two maps $R[\lambda]$ and $R[\lambda']$ have the *same combinatorics up to level n* if $\lambda, \lambda' \in \Lambda^n$. If two maps $R[\lambda]$ and $R[\lambda']$ have the same combinatorics up to level n, a *pseudo-conjugacy* (up to level n) between $R[\lambda]$ and $R[\lambda']$ is an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $H \circ R[\lambda] = R[\lambda'] \circ H$ everywhere outside int U^n.

5.2. *A priori bounds in parameter space.* In this section we prove the following theorem giving us *a priori* bounds in the principal nest of parapuzzle pieces.

THEOREM 5.1. *Let $\{f_\lambda\}$ be a non-trivial, real analytic family of unimodal maps. Suppose that $f \in \{f_\lambda\}$ is a Yoccoz map with recurrent critical point and let $\{n_k\}_{k=1}^\infty$ be the sequence of non-central levels in its principal nest. There exists $\delta > 0$ such that:*

- $|I^{n_k}|/|I^{n_k+1}| > 1+\delta$;
- $|\Lambda_{\mathbb{R}}^{n_k}|/|\Lambda_{\mathbb{R}}^{n_k+1}| > 1+\delta$;

where $I^0 \supset I^1 \supset I^2 \supset \cdots$ is the principal nest with $I^0 = J$ a sufficiently small nice interval about the critical point of f, and $\Lambda_{\mathbb{R}}^1 \supset \Lambda_{\mathbb{R}}^2 \supset \Lambda_{\mathbb{R}}^3 \supset \cdots$ is the real principal nest of parapuzzle pieces for f.

We divide the proof of this result into two cases: when the map has sufficiently big geometry and when its post-critical set is persistently recurrent. When the map has sufficiently big geometry this is a consequence of [**6**, Lemmas 6.12 and 6.13].

PROPOSITION 5.1. *There exists a constant C_2 with the following property. Suppose that f_0 is a non-renormalizable Yoccoz map with the property that for some sufficiently small critical puzzle piece Y, $|Y|/|\mathcal{L}_0(Y)| > C_2$. Then there exists a constant $\delta > 0$, such that if $\{n_k\}_{k=1}^{\infty}$ denotes the subsequence of levels in the principal nest corresponding to non-central returns of the critical point to I^{n_k}, where we take $I^0 = Y$, then $|\Lambda_{\mathbb{R}}^{n_k}|/|\Lambda_{\mathbb{R}}^{n_k+1}| > 1+\delta$.*

Before treating the case of maps with bounded geometry, we need a few preparatory lemmas.

LEMMA 5.1. [**31**] *Suppose that $\delta > 0$ and $U \Subset U' \neq \mathbb{C}$ are disjoint topological discs with the property that* $\mathrm{mod}(U' \backslash U) > \delta$. *Then there exists a constant $c = c(\delta)$ such that*

$$d(z, \partial U') \geq c\, \mathrm{diam}(U)$$

for any $z \in \overline{U}$, where distance and diameter are measured in the Euclidean metric.

The following lemma is a complex analogue of Proposition 8.1, 2. of [**13**].

LEMMA 5.2. *Suppose that $f : \bigcup_j U_j \to U$ is a non-renormalizable puzzle map with a persistently recurrent critical point. Set $E^0 = U$ and let $E^0 \supset E^1 \supset E^2 \supset \cdots$ be its enhanced nest (see Proposition 3.2). For any $C > 0$, there exists a constant $C' > 0$ such that if the critical orbit passes through a component of V of* $\mathrm{Dom}(R_{E^n})$ *such that* $\mathrm{mod}(E^n \backslash V) > C'$, *then* $\mathrm{mod}(E^{n+1} \backslash \mathcal{L}_0(E^{n+1})) > C$.

Proof. Let us first prove the following.

CLAIM. *For any $\rho > 0$, there exists $\rho' > 0$ such that if* $\mathrm{mod}(E^n \backslash E^{n+1}) > \rho$, *then E^{n+1} is ρ'-nice. Moreover, $\rho' \to \infty$ as $\rho \to \infty$.*

Proof. Consider the associated real puzzle pieces $E_{\mathbb{R}}^n := E^n \cap \mathbb{R}$. It follows from Proposition 3.2 that there exists a constant η such that $\eta \to \infty$ as $\rho \to \infty$ and if $\mathrm{mod}(E^n \backslash E^{n+1}) > \rho$, then $(1+2\eta)E_{\mathbb{R}}^{n+1} \subset E_{\mathbb{R}}^n$. By [**13**, Lemma 9.7], there exists a constant $\eta' > 0$ such that if J is any component of the return map to E^{n+1} intersecting the post-critical set, then $(1+2\eta')J \subset E_{\mathbb{R}}^{n+1}$, and $\eta' \to \infty$ as $\eta \to \infty$. Now, by Proposition 3.2, we have that there exists $\rho' > 0$ such that E^{n+1} is ρ'-nice and $\rho' \to \infty$ as $\rho \to \infty$. That completes the proof of the claim. □

Thus, to complete the proof of the lemma, it suffices to show that $\mathrm{mod}(E^n \backslash E^{n+1})$ is large when C' is large. Let $z \in \omega(0) \cap E^n$, set $V = \mathcal{L}_z(E^n)$ and assume that

$\mathrm{mod}(E^n\backslash V) > C'$. If $0 \in V$, then, since $E^{n+1} \subset V$, we are done. So assume that $0 \notin V$. Since E^n is δ-fat, $\mathrm{dist}_{E^n}(V, \partial E^n)$ is bounded from below. Let $V' = \mathcal{L}_z(\Gamma(\mathcal{A}(E^n)))$. Then $V \supset V'$. Since $\mathcal{L}_0(V') \supset \Gamma^2(\mathcal{A}(E^n)) \supset E^{n+1}$, it follows that $\mathrm{mod}(E^n\backslash E^{n+1})$ is large. □

As an immediate corollary we have the following.

COROLLARY 5.1. *Suppose that f is a non-renormalizable R-map with geometry bounded by M. There exist constants $\alpha, \alpha' > 0$ depending on M such that if V is a component of* $\mathrm{Dom}(R_{E^n})$ *such that $\omega(0) \cap V \neq \emptyset$, then* $\mathrm{mod}(E^n\backslash V) < \alpha$ *and* $\mathrm{diam}\, V \geq \alpha' \,\mathrm{diam}\, E^n$.

PROPOSITION 5.2. *Suppose that f_0 is a non-renormalizable Yoccoz map with bounded geometry in a non-trivial analytic family, $\{f_\lambda\}$. Then there exists a constant $\delta > 0$ such that $|\Lambda_{\mathbb{R}}^{n_k}|/|\Lambda_{\mathbb{R}}^{n_k+1}| > 1 + \delta$ for all k sufficiently large.*

Proof. Let v be a vector field transverse to $\mathcal{H}_{f_0}$. The existence of such a vector field was established in the proof of Theorem 4.2. Consider the one-parameter analytic family $\Lambda = \{f_{0+\lambda v} : \lambda \in \mathbb{D}\} = f_0 \circ (\mathrm{id} + \lambda v)$, $\lambda \in \mathbb{D}$. Since v is transverse to $\mathcal{H}_{f_0}$, Λ is non-trivial.

By Proposition 3.6, there exists a real critical puzzle piece J, which we may take to be from the real enhanced nest for f_0, so that R_J extends to a complex puzzle map which we will denote by $f_0 : \bigcup U_j \to U$. Starting with $E^0 = U$, let $E^0 \supset E^1 \supset E^2 \supset \cdots$ be the enhanced nest for f_0 (see Proposition 3.2). Let α be the constant from Corollary 5.1. Let $\mathcal{E}^n$ be the collection of components, E^n_j, of $\mathrm{Dom}(R_{E^n})$ such that $\mathrm{mod}(E^n\backslash E^n_j) < \alpha$. Because f_0 has bounded geometry, we know that $\mathcal{E}^n$ includes all the components of $\mathrm{Dom}(R_{E^n})$ that intersect $\omega(c)$, and it may also contain finitely many additional components with large diameters. We define a modified generalized renormalization operator $\tilde{R} : \mathcal{A}_a \to \mathcal{B}_{\mathrm{nor}}(\mathcal{E}^n)$ as the map that sends f to $R_{E^n}|\mathcal{E}^n$, a generalized polynomial-like map. Notice that Lemmas 4.2 and 4.3 still hold if we include these finitely many more components in the domains of the renormalized maps. We require this additional care to avoid discarding combinatorial classes of maps whose critical orbits pass through large components of the domain of the return map to E^n, which do not intersect $\omega_{f_0}(0)$.

It follows from Proposition 3.6 that there exists a neighbourhood B of f_0 in $\mathcal{A}_a$ such that the puzzle map associated to E^n persists over B.

Let $g_n \in \mathcal{B}_{\mathrm{nor}}(\mathcal{E}^n)$ be the image of $\tilde{R} f_0$. By Lemma 4.3, $\tilde{R}$ maps a neighbourhood B' of f_0 to a neighbourhood of g_n. Let $\tilde{B} = \tilde{R}(B' \cap B)$. The puzzle pieces for the generalized polynomial-like map g_n persist over $\tilde{B}$. Hence we get a tube map $\boldsymbol{g}_n : \boldsymbol{\mathcal{E}}^n \to \boldsymbol{E}^n$ with generalized polynomial-like maps $g_n[\lambda] : \mathcal{E}^n[\lambda] \to E^n[\lambda]$ as fibres. Let h_n be the holomorphic motion of the puzzle pieces $E^n \cup \mathcal{E}^n$.

By Corollary 5.1 and Proposition 3.2 we have that there exist constants $\alpha > 0$ and $\alpha' > 0$ such that for any $V \in \mathcal{E}^n$, $\mathrm{mod}(E^n\backslash V) > \alpha$ and $\mathrm{diam}\, V > \alpha' \,\mathrm{diam}\, E^n$. This implies that there exists a constant $c > 0$ such that $d(\partial E^n, V) > c \,\mathrm{diam}\, E^n$. Notice that c is independent of the level n of the enhanced nest. By the λ-lemma, we can choose a neighbourhood of g_n so that dilatation of the holomorphic motion of the puzzle is close to 1. Hence there exists a neighbourhood $\tilde{B}'$ of g_n, so that, provided that $g_n[\lambda]$ is pseudo-conjugate to g_n outside of E^{m+1}, the puzzle pieces ∂E^{m+1}, $\mathcal{E}^m$ all move holomorphically over $\tilde{B}'$.

Let $C = \{g_n[\lambda](0) : \lambda \in \tilde{B}'\}$. This gives us a neighbourhood of the critical value of g_n. For each $m > n$, let E^m_L be the pullback of E^m containing the critical value, $g_n(0)$, under the first landing map to E^m. By Proposition 3.2, there exists $m > n$ such that $E^m_L \subset \{g_n[\lambda] : \lambda \in \tilde{B}'\}$. Let $\tilde{B}'' \subset \tilde{B}'$ be such that $E^m_L = \{g_n[\lambda] : \lambda \in \tilde{B}''\}$. Set $\tilde{\mathcal{V}}^0 = \tilde{B}''$. This gives us the start of the construction for the principal nest of parapuzzle pieces about the generalized polynomial-like map g_n.

As with first return maps, we can lift the R-family, $\boldsymbol{g}_n : \boldsymbol{\mathcal{E}}^n \to \boldsymbol{E}^n$, to an R-family $\boldsymbol{g}_{n+1} : \boldsymbol{\mathcal{E}}^{n+1} \to \boldsymbol{E}^{n+1}$. Let $\underline{d} = (j_1, j_2, \ldots, j_k)$. We define

$$\boldsymbol{E}^n_{\underline{d}} = \{(\lambda, z) \in \boldsymbol{E}^n : g^k_n[\lambda](z) \in E^n_k[\lambda]\} \quad \text{and} \quad \boldsymbol{g}^{\underline{d}}_n = \boldsymbol{g}^{|\underline{d}|}_n | \boldsymbol{E}^n_{\underline{d}}.$$

Let $\boldsymbol{W}^n_{\underline{d}}$ be the pullback of $\boldsymbol{E}^{n+1}$ by $\boldsymbol{g}^{\underline{d}}_n$. We denote the associated landing map by $L_E(\boldsymbol{g}_n) : \bigcup \boldsymbol{W}^n_{\underline{d}} \to \boldsymbol{E}^{n+1}$. We define a holomorphic motion $L_E(h_n)$ as follows. The leaf through $z \in \partial E^n$ is the leaf of h through z, and if there is a smallest $E^n_{\underline{d}}$ such that $z \in E^n_{\underline{d}}$, we let the leaf through z be the pullback of the leaf through $g^{|\underline{d}|}_n(z)$ of h_n by $\boldsymbol{g}^{\underline{d}}_n$. We extend $L_E(h_n)$ to E^n using the extension lemma. Let $\underline{d}_0$ be the sequence of components of $\mathcal{E}^n$ through which the critical value travels before landing in E^{n+1}. We obtain an R-family $(\boldsymbol{g}_{\boldsymbol{n+1}}, h_{n+1})$, where $\boldsymbol{g}_{n+1}$ is the return map to $\boldsymbol{E}^{n+1}$ through iterates of $\boldsymbol{g}_n$ and h_{n+1} is the lift of $L_E(h_n)$ by $(\boldsymbol{F}^n | \boldsymbol{E}^{n+1}, \tilde{\mathcal{V}}^0, W^n_{\underline{d}})$, where $\boldsymbol{F}^n$ is the first landing map to $\boldsymbol{E}^n$, wherever it is defined. We have that the components of $\mathcal{E}^m$ move holomorphically over $\tilde{\mathcal{V}}^0$, and we have a tube map $\boldsymbol{g}_m : \boldsymbol{\mathcal{E}}^m \to \boldsymbol{E}^m$ with the map $\lambda \mapsto g_m[\lambda](0)$, $\lambda \in \tilde{\mathcal{V}}^0$, as a diagonal. Let χ be the associated holonomy family. Note that the holomorphic motion of the puzzle $E^m \cup \mathcal{E}^{m-1}$ is K-qc with K controlled by the bounds for the enhanced nest.

Set $V^0 = E^m$ and $V^1 = \mathcal{L}_0(E^m)$. Let $\Lambda^0 = \chi(V^0)$. If $V^0 \supset V^1 \supset V^2 \supset \cdots$ is the principle nest for g_0, we let Λ^i be the corresponding parapuzzle pieces, as constructed in §5.1. The fact that holonomy family is K-qc lets us transfer the bounds, $\operatorname{mod}(V^{n_k} \backslash V^{n_k+1}) > \delta$, where the sequence n_k is the sequence of non-central returns in the principal nest, in the dynamical plane to the parameter plane: there exists $\delta' > 0$ such that $\operatorname{mod}(\Lambda^{n_k} \backslash \Lambda^{n_k+1}) > \delta'$. The proof of this estimate is identical to the proof of Theorem 4.5 of [**3**] where the same result was proved when f is a unicritical polynomial.

The analyticity of the generalized renormalization operator allows us to transfer the bounds in the family of generalized polynomial-like maps to a neighbourhood of f_0 in the family $\{f_{0+\lambda v} : \lambda \in \mathbb{D}\}$. Let $\mathcal{V}^1 \supset \mathcal{V}^2 \supset \mathcal{V}^3 \supset \cdots$ be the resulting nest of parapuzzle pieces about f. Let $\Lambda^n_{\mathbb{R}} = \{\mathcal{H}_g \cap \{f_\lambda\} : g \in \mathcal{V}^n\}$. Since the lamination near f in $\mathcal{U}_a$ by the topological conjugacy classes is transversally quasiconformal, its restriction to the real slice is quasisymmetric, so we can transfer the bounds obtained above to a neighbourhood of f_0 in $\{f_\lambda\}$ to get that there exists a constant $\delta'' > 0$ such that$|\Lambda^{n_k}_{\mathbb{R}}|/|\Lambda^{n_k+1}_{\mathbb{R}}| > 1 + \delta''$. $\square$

6. *Conclusion*

Let $\mathcal{F}$ denote the set of all finitely renormalizable non-regular parameters. By Sand's theorem [**26**], the set maps with recurrent critical point have full measure in $\mathcal{F}$.

Let C be the maximum of the constants C_1 and C_2 from Propositions 4.4 and 5.1, respectively. Suppose that $f \in \mathcal{F}$ has a recurrent critical point and let g be its last renormalization. We say that f has sufficiently big geometry if there exists a critical real puzzle piece J for g such that such that $|J|/|\mathcal{L}_0(J)| > C$. If f does not have sufficiently big geometry, then it obviously has bounded geometry. Hence, by Proposition 2.1 it has a persistently recurrent critical point. In either case we have that Theorem A and Theorem 5.1 hold for f.

Let us remark on how we complete the proof of Theorem B. Let $\mathcal{S}$ denote the set of at most finitely renormalizable maps f for which there exist constants $C > 0$ and $\lambda > 1$ such that the following holds. Let g be the last renormalization of f. Let $I^0 \supset I^1 \supset I^2 \supset \cdots$ be the principal nest for g and let s_n denote the return time of the critical point to I^n. We require that:

- $|I^{n-1}|/|I^n| > C\lambda^n$;
- $|\Lambda_{\mathbb{R}}^{n-1}|/|\Lambda_{\mathbb{R}}^n| > C\lambda^n$;
- $s_n > Cn$.

Theorem 5.1 together with Proposition 3.1 implies by the real analogue of [**3**, Lemma 5.3] (see [**3**, §5.2]) that $\mathcal{S}$ has full measure in the space of finitely renormalizable maps. By [**5**, Remark 2.1], the statistical argument carried out in that paper applies to $\mathcal{S}$. Hence we have the following theorem.

THEOREM 6.1. [**6**, Theorems C and E] *Suppose that $\{f_\lambda\}$ is a non-trivial analytic family of unimodal maps such that the set of infinitely renormalizable maps in the family has measure zero. Then almost every $f_\lambda \in \{f_\lambda\}$ is regular or Collet–Eckmann.*

Additionally, the argument of [**6**, §9] applies in our setting, and implies the following theorem.

THEOREM 6.2. [**6**, Theorem B] *Suppose that $\{f_\lambda\}$ is a non-trivial analytic family of unimodal maps such that the set of infinitely renormalizable maps in the family has measure zero. Then almost every $f_\lambda \in \{f_\lambda\}$ is either regular or has a renormalization that is topologically conjugate to a polynomial.*

These two results imply our Theorem B, and indeed more is true. The results of this paper imply that the arguments of [**7**] go through in higher degrees as well.

THEOREM 6.3. [**6, 7**] *Suppose that $\{f_\lambda\}$ is a non-trivial analytic family of unimodal maps such that the set of infinitely renormalizable maps in the family has measure zero. Then almost every $f_\lambda \in \{f_\lambda\}$ which is non-regular is Collet–Eckmann and satisfies:*

(1) *the critical point is polynomially recurrent with exponent* 1,

$$\limsup \frac{-\ln|f_\lambda^n(0)|}{\ln n} = 1;$$

(2) *the critical orbit is equidistributed with respect to the absolutely continuous invariant measure μ,*

$$\lim \frac{1}{n}\sum_{i=0}^{n-1} \phi(f_\lambda^i(0)) = \int \phi \, d\mu,$$

for any continuous function $\phi : I \to \mathbb{R}$;

(3) *the Lyapunov exponent of the critical value,* $\lim(1/n)\ln|Df_\lambda^n(f_\lambda(0))|$, *exists and coincides with the Lyapunov exponent of μ;*

(4) *the Lyapunov exponent of any periodic point p contained in* $\operatorname{supp}\mu$ *is determined (via an explicit formula) by combinatorics (more precisely, by the itineraries of p and of the critical point).*

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